The similarity transformation

In our earlier article, we treated the change of basis. Now, one may ask: “How would the matrix change, under the change of basis?” If you closely think about our articles “Matrices and Linear Alegbra” and ”The mathematical definition of vector,” you will see that a matrix is a linear map (i.e. linear operator) from a vector to a vector. If vectors are represented in a different basis, the matrix should be as well. For example, if a matrix in a certain basis satisfies:

$$Av = u \tag{1}$$

where A is $n \times n$ matrix, v and u are $n \times 1$ matrices, (i.e. the components of n-dimensional vectors) then, in another basis, the corresponding matrix should satisfy following:

$$A'v' = u' \tag{2}$$

where

$$v' = Pv, \qquad u' = Pu \tag{3}$$

for a suitable $n \times n$ matrix P that describes the change of basis. For example, in our earlier example in the article “change of basis,” we had

$$v = \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} v'_x \\ v'_y \end{pmatrix} = Mv' \tag{4}$$

for $M = P^{-1}$ which follows from (3). Again, I want to emphasize that (1) and (2) describe the same situation. They are same vectors and same matrix, even though the explicit values are different as the basis is different.

Now, let’s closely examine our earlier equations:

$$\begin{aligned} A'v' &= u' && (5) \\ A'(Pv) &= (Pu) && (6) \\ A'(Pv) &= P(Av) && (7) \\ A'P &= PA && (8) \\ (A'P)P^{-1} &= (PA)P^{-1} && (9) \\ A' &= PAP^{-1} && (10) \end{aligned}$$

So, we have shown how matrices transform under the change of basis. This is called "similarity transformation." Now, notice that eigenvalues of A' should be the same as eigenvalues of A, since they are both equally good descriptions of the same linear map, and the linear map gives the eigenvectors and eigenvalues "basis-independent," We can also explicitly check this: For an eigenvector e with eigenvalue λ, we have:

$$Ae = \lambda e \tag{11}$$
$$PAe = \lambda Pe \tag{12}$$
$$A'Pe = \lambda Pe \tag{13}$$

where from the second line to the third line we used (8). Thus, we conclude Pe is the eigenvector of A' with the eigenvalue λ. In the notation of (3), $e' = Pe$ is the eigenvector of A'. Notice that this is the same eigenvector just in a different basis.

Let's consider now the matrix multiplication. Consider two matrices A and B. In the "primed"-basis, they are expressed as $A' = PAP^{-1}$ and $B' = PBP^{-1}$. Then, how should AB be expressed in the primed-basis? It should be $P(AB)P^{-1} = PABP^{-1}$, as this is the law for similarity transformation. On the other hand, from the point of view of primed-basis, this should be $A'B'$ since it represents multiplication of two linear operators in primed-basis. Let's check that they are same.

$$A'B' = PAP^{-1}PBP^{-1} = PAIBP^{-1} = PABP^{-1} \tag{14}$$

They are indeed same.

Problem 1. Express the inverse of $A' = PAP^{-1}$ in terms of A^{-1}, P, P^{-1} and check that you have obtained the correct answer by multiplying A' and its inverse to get the identity matrix.

Now, let's ask how the determinant would transform under similarity transformation. Remember the following theorem in our last article:

$$\det(AB) = \det A \det B \tag{15}$$

Then, we can say:

$$\det A' = \det(PAP^{-1}) = \det P \det A \det P^{-1} \tag{16}$$
$$= \det P \det P^{-1} \det A = \det(PP^{-1}) \det A \tag{17}$$
$$= \det I \det A = \det A \tag{18}$$
$$\tag{19}$$

So, the determinant is invariant under similarity transformation!

Now, let's see how trace transforms.

$$\mathrm{tr}A' = \mathrm{tr}(PAP^{-1}) = \mathrm{tr}(P^{-1}PA) = \mathrm{tr}(A) \tag{20}$$

So, the trace is also invariant under similarity transformation!

Therefore, we conclude that eigenvalues, determinant, and trace are "basis-free" intrinsic properties of a linear map. In the next article, we will see how the determinant and the trace can be expressed in terms of eigenvalues.

Summary

- Similarity transformation is given by $A' = PAP^{-1}$.
- Eigenvalues, determinant and trace are "basis-free" intrinisic properties of a linear map. They don't change under similarity transformation.