The Sandwich theorem and the limit of $\sin\theta/\theta$

In this article, we will first explain the sandwich theorem, then apply it to calculate $\lim_{\theta\to 0}\sin\theta/\theta$.

The Sandwich theorem says, if

$$f(x) \le g(x) \le h(x) \tag{1}$$

is satisfied, and we have

$$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L \tag{2}$$

Then,

$$\lim_{x\to a} g(x) = L \tag{3}$$

Even though the rigorous proof may be hard, the theorem is intuitively clear. Let me explain why. (1) implies that $g(x)$ is "sandwiched" between $f(x)$ and $h(x)$. Thus, its limit must be sandwiched between the limits of $f(x)$ and $h(x)$. In other words,

$$\lim_{x\to a} f(x) \le \lim_{x\to a} g(x) \le \lim_{x\to c} h(x) \tag{4}$$

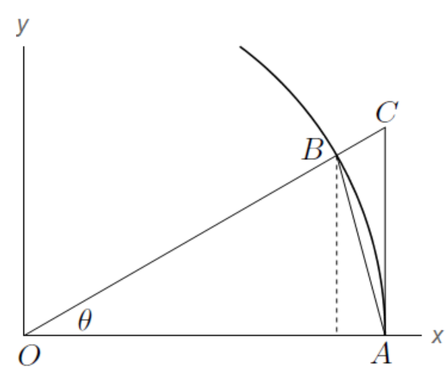
However, from (2), we have

$$L \le \lim_{x\to a} g(x) \le L \tag{5}$$

Thus, we conclude (3). In other words, the function $f(x)$ and $h(x)$ sandwich the function $g(x)$ so tightly that the limits of $f(x)$ and $h(x)$ determine the limit of $g(x)$.

Problem 1. (Hint[1])

$$\lim_{x\to 0} x\sin\frac{1}{x} =? \tag{6}$$



[1]Use $-x \le x\sin\frac{1}{x} \le x$ for $x > 0$ and $x \le x\sin\frac{1}{x} \le -x$ for $x < 0$.

Now, let's calculate $\lim_{\theta\to 0}\sin\theta/\theta$. See the figure. You see that a circle centered around the origin O, ΔOAC, ΔOAB. We will compare the area of ΔOAB, the sector OAB and ΔOAC. The radius of the circle is given by r and the angle AOB by θ. We use the radian here. From the figure, the following relation is clear.

$$\text{Area of } \Delta OAB < \text{Area of sector } OAB < \text{Area of } \Delta OAC \tag{7}$$

The base of ΔOAB is the length of $\overline{O}A$, which is r. The height of ΔOAB is given by the length of dotted line. As the length of $\overline{OB}$ is r, the height of ΔOAB is $r\sin\theta$. The base of ΔOAC is the length of $\overline{OA}$, which is r. The height of ΔOAC is given by the length of $\overline{AC}$ is $r\tan\theta$. Then, (7) yields

$$\frac{1}{2}r\cdot r\sin\theta < \frac{1}{2}r^2\theta < \frac{1}{2}r\cdot r\tan\theta \tag{8}$$

Dividing all three terms by $(r^2\sin\theta)/2$, we get

$$1 < \frac{\theta}{\sin\theta} < \cos\theta \tag{9}$$

We can now use the Sandwich theorem. The above relation implies

$$\lim_{\theta\to 0} 1 \leq \lim_{\theta\to 0}\frac{\theta}{\sin\theta} \leq \lim_{\theta\to 0}\cos\theta \tag{10}$$

$$1 \leq \lim_{\theta\to 0}\frac{\theta}{\sin\theta} \leq 1 \tag{11}$$

which implies the following by the Sandwich theorem

$$\lim_{\theta\to 0}\frac{\theta}{\sin\theta} = 1 \tag{12}$$

Thus, we conclude

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1 \tag{13}$$

Problem 2. (Hint[2])

$$\lim_{\theta\to 0}\frac{\tan\theta}{\theta} =? \tag{14}$$

Problem 3. (Hint[3])

$$\lim_{\theta\to 0}\frac{\sin 3\theta}{\theta} =? \tag{15}$$

Summary

- The Sandwich theorem says, if

  $$f(x) \leq g(x) \leq h(x), \qquad \text{and} \qquad \lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$$

  then,

  $$\lim_{x\to a} g(x) = L$$

- Using the sandwich theorem, we can prove

  $$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

[2] Use $\tan\theta = \sin\theta/\cos\theta$.

[3] Let $3\theta = \alpha$, use $\sin 3\theta/\theta = 3\sin\alpha/\alpha$.