Solid angle

What is the definition of angle in the unit of radian? It is the arc length divided by the radius. See Fig. 1. It is given by $\theta = l/r$. Angle is one-dimensional in two-dimensional space, as the arc is one-dimensional and can be drawn in two-dimensional space. Then, can we similarly define the two-dimensional angle in three-dimensional space? Yes, see Fig. 2. The two-dimensional angle is called "solid angle," and often denoted by Ω. It is given by $\Omega = A/r^2$, where A is the relevant subtended area on the sphere.

So, why should we divide the area by the radius squared instead of the radius as was the case with angle? In case of angle, if we double the radius the arc length is doubled, as long as we have the same angle. In case of the solid angle, if we double the radius, the relevant subtended area is quadrupled, as long as we have the same solid angle; area is always proportional to the length squared. If the shape remains same, but the length is doubled, the area is quadrupled.

You can check this again as follows. If we have the same angle, the ratio of the arc length to the circumference must not depend on the radius. $r\theta/(2\pi r) = \theta/(2\pi)$. It is indeed. If we have the same angle, the ratio of the subtended area to the area of sphere must not depend on the radius. $r^2\Omega/(4\pi r^2) = \Omega/(4\pi)$. It is indeed.

You can think along this way as well. Both angle and solid angle must be dimensionless. As the arc length has the dimension of length, to get a dimensionless ratio, you need to divide it by the radius. As the subtended area has the dimension of length squared, to get a dimensionless ratio, you need to divide it by the radius squared.

Finally, what is the full angle? It is 2π (i.e. $360°$). It is the circumference divided by the radius. ($2\pi r/r = 2\pi$) What is the full solid angle? It is 4π. It is the sphere area divided by the radius squared. ($4\pi r^2/r^2 = 4\pi$)

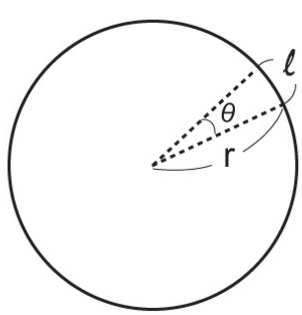


Figure 1: angle

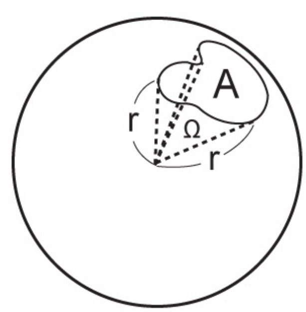
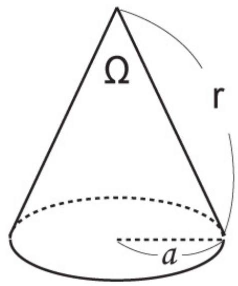


Figure 2: solid angle

Problem 1. You see a right circular cone in the figure. The base is a circle with radius a, and the side has length r. What is the solid angle Ω? (Hint[1])



Summary

- As an angle is defined by $\theta = l/r$ where l is the length of the arc and r is the radius, a solid angle is defined by $\Omega = A/r^2$, where A is the subtended area on a sphere.
- The whole solid angle is 4π, as the area of sphere is $4\pi r^2$.

[1] Perform the integration which we did with Figure 4 in the last article.