Spherical coordinate system, the surface area of sphere, and the volume of ball

In this article, we will deal with the spherical coordinate system, which is widely used in physics when there is a spherical symmetry. For our purpose, it will be useful in solving hydrogen atom problem later; A hydrogen atom is composed of a proton and an electron, and the Coulomb potential of the electron due to the presence of proton only depends on the distance to the proton and not on the direction to the proton. It is given by "$-ke^2/r$," where r is the distance. Therefore, there is a spherical symmetry (i.e. independence from the direction) for the potential energy of electron. In this article, we will use the spherical coordinate system to obtain the formulas for the surface area of sphere and the volume of ball, which most high school students learn, but without their derivation.

See Fig. 1. We have denoted spherical coordinate system. Instead of x, y, and z to denote a position in 3-dimensional space, we have r, θ and ϕ. Notice that the distance from the origin is r. Notice also that the right triangle, which is denoted by dotted line, has a height of $r\cos\theta$ and a base of $r\sin\theta$. This suggests the following relation between x,y,z and r,θ,ϕ.

$$x = r\sin\theta\cos\phi \tag{1}$$

$$y = r\sin\theta\sin\phi \tag{2}$$

$$z = r\cos\theta \tag{3}$$

where $r \geq 0$, $0 \leq \theta \leq \pi$ and $0 \leq \phi < 2\pi$. Notice again that the Coulomb potential an

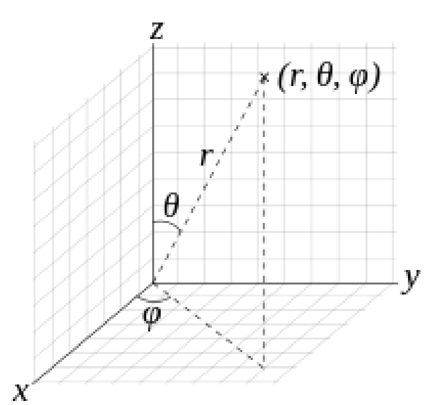

Figure 1: Spherical coordinate

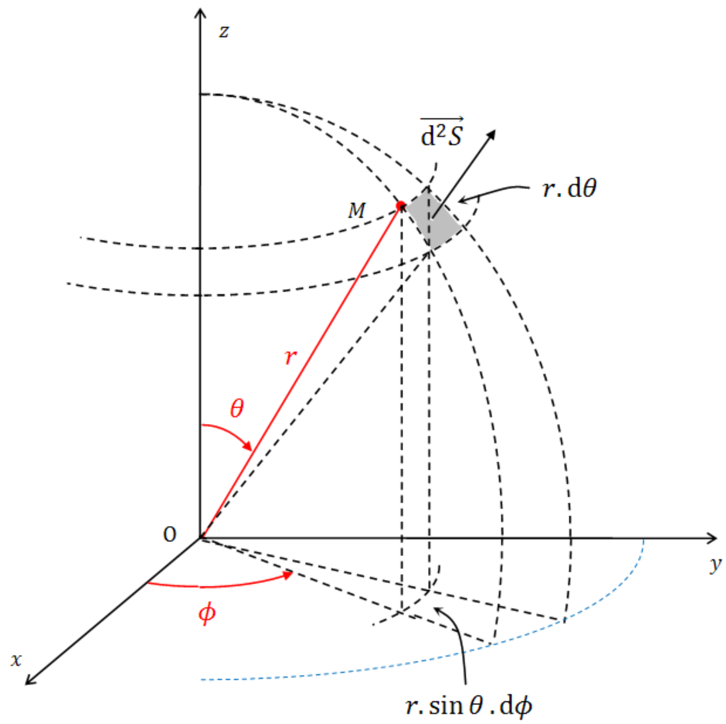

Figure 2: Area element

electron has in the hydrogen atom has the spherical symmetry, because it only depends on r, but not on θ or ϕ.

Given this, let's apply spherical coordinate to obtain the surface area of a sphere with radius r. See Fig. 2. The shaded region is an infinitesimal part of the surface of the sphere. It is a rectangle and the area is given by $(r\sin\theta d\phi)\cdot(rd\theta)$. If we add all these up for different θ and ϕ, we will get the surface area of sphere. Let's do this:

$$A = \int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}(r\sin\theta d\phi)\cdot(rd\theta) = \int_{\theta=0}^{\pi} r^2\sin\theta d\theta 2\pi = 2\pi r^2(-\cos\pi+\cos 0) = 4\pi r^2 \tag{4}$$

So, this is the surface area of the sphere. Now, what is the enclosed volume of the sphere? The inner region of a sphere is called a "ball." So what is the volume of ball with radius R? See Fig.3. We see that the infinitesimal volume area is given as follows:

$$dV = (r\sin\theta d\phi)\cdot(rd\theta)\cdot dr \tag{5}$$

Integrating, we get:

$$V = \int_{r=0}^{r=R}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} dV = \int_{r=0}^{r=R}\int_{\theta=0}^{\pi} dr(r^2\sin\theta d\theta 2\pi) \tag{6}$$

$$= \int_{r=0}^{r=R} 4\pi r^2 dr = \frac{4}{3}\pi R^3 \tag{7}$$

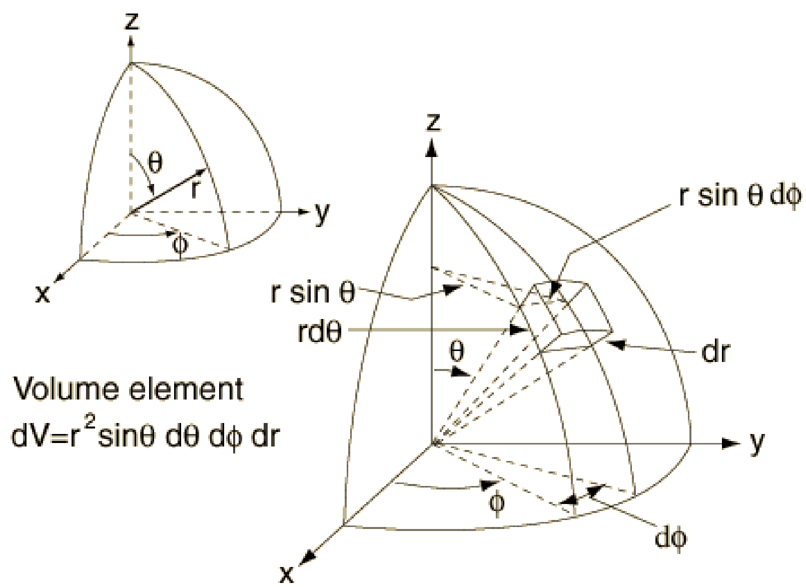

Figure 3: Volume element

where from the first line to the second line we used (4).

Actually, we could have obtained it by a slightly different method, using Jacobian as follows:

$$V = \int\int\int dV = \int\int\int dxdydz = \int\int\int J dr d\theta d\phi \tag{8}$$

where J is the Jacobian of (1∼3). A tedious calculation shows that it is $r^2 \sin\theta$, thus agreeing with (5). Then, our formula reduces to (6), thus obtaining the same result.

(Figure 2 is reproduced from `http://commons.wikimedia.org/wiki/File:Coordonn%C3%A9es_sph%C3%A9riques_04.png`)

Summary

- Spherical coordinate system is described by the distance to the origin r, and two angles θ, ϕ.
- $z = r\cos\theta, \qquad x = r\sin\theta\cos\phi, \qquad y = r\sin\theta\sin\phi$