# Why are string theories only consistent in certain dimensions?

There are many ways to show that the bosonic string theory (string theory only with bosons) is consistent only in 26 spacetime dimensions, and the superstring theory (string theory with both bosons and fermions) is consistent only in 10 spacetime dimensions, but one way is ensuring that there is no "negative-normed" physical state. In this article, I will explain what it is.

We know that a probability must be always non-negative. Let's see what consequences this requirement implies. Suppose that a physical state (i.e., a state that actually occurs in our nature) is given by $|\text{phys}\rangle$. Then, as we have seen, $\langle\text{phys}|i\rangle\langle i|\text{phys}\rangle$ represents the probability that $|\text{phys}\rangle$ becomes $|i\rangle$ upon observation. Suppose $|i\rangle$s form a complete basis of the Hilbert space (i.e., they span the entire Hilbert space). In other words, no vector in the Hilbert space is "left out." Given this, let's add all the probabilities. The total probability is given by

$$\text{total probability} = \sum_i \langle\text{phys}|i\rangle\langle i|\text{phys}\rangle = \langle\text{phys}|\text{phys}\rangle \tag{1}$$

where we used the completeness relation. As the total probability is always positive we conclude that $\langle\text{phys}|\text{phys}\rangle > 0$ must be satisifed, if $|\text{phys}\rangle$ is to represent an actual, physical state. There may exist states $|\chi\rangle$ that satsify (so-called "negative-norm states")

$$\langle\chi|\chi\rangle < 0 \tag{2}$$

but such a state must not be physical (i.e., it must not correspond to a state that actually occurs in our nature). In string theory, a physical state is defined by

$$(L_0 - a)|\text{phys}\rangle = L_1|\text{phys}\rangle = L_2|\text{phys}\rangle = L_3|\text{phys}\rangle = \cdots = 0 \tag{3}$$

where $a$ is a certain number, and $L_0, L_1, L_2, \cdots$ are matrices called "Virasoro generators" of which the exact definition is not a concern of this article.

There are also states called "spurious states." A spurious state $|\text{spur}\rangle$ is a physical state that satisfy the following condition for all physical states $|\text{phys}\rangle$.

$$\langle\text{phys}|\text{spur}\rangle = 0 \tag{4}$$

What consequences does this condition imply? It implies that

$$|\text{phys}\rangle + c|\text{spur}\rangle \tag{5}$$

describes the same state as (where $c$ is an arbitrary number)

$$|\text{phys}\rangle \tag{6}$$

Why? What we actually measure in experiments are not the wave functions or the state vectors but probabilities, which in turn are given only in terms of inner products between two state vectors. Since we can only measure probabilities for transitions from one *physical* state to another *physical* state, all that matters is the inner product between *physical* states. Let's consider an inner product between a physical state 1 $|\text{phys}_1\rangle$ and a physical state 2$|\text{phys}_2\rangle$. Then, we have

$$\langle\text{phys}_2|\left(|\text{phys}_1\rangle + c|\text{spur}\rangle\right) = \langle\text{phys}_2|\text{phys}_1\rangle \tag{7}$$

Thus, the two vectors $(|\text{phys}_1\rangle + c|\text{spur}\rangle)$ and $|\text{phys}_1\rangle$ describe the same state. Those of you who know equivalence relation will understand what I mean if I write

$$|\text{phys}\rangle \sim |\text{phys}\rangle + c|\text{spur}\rangle \tag{8}$$

Anyhow, the analysis of spurious states is crucial to ensure that no physical states have negative norms (i.e., that $\langle\text{phys}|\text{phys}\rangle \geq 0$). This leads to the determination of spacetime dimensions in which string theories are consistent.

**Problem 1.** Explain why a physical state of the form (5) satisfies the physical state condition (3). (Hint[1])

## Summary

- String theories are only consistent in certain spacetime dimensions.
- Ensuring that there are no negative-normed physical states is one of the way to determine these dimensions.

[1]Use the condition that a spurious state is a physical state.