$SU(2)$ Lie group and Lie algebra

As mentioned in an earlier article, "What is a gauge theory?," $SU(N)$ is the group of $N \times N$ unitary matrices with determinant 1. In this article, we will focus on the $SU(2)$ group and its Lie algebra, as it is the simplest non-Abelian group and plays a pivotal role in loop quantum gravity.

Now, let's try to describe U, an element of $SU(2)$, explicitly as follows.

$$U = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{1}$$

Since a unitary matrix satisfies $U^\dagger = U^{-1}$, we have:

$$\begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \frac{1}{\det U} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{2}$$

Recalling that $\det U = 1$, we conclude $a^* = d$, $b^* = -c$. Therefore, we have:

$$U = \begin{pmatrix} a & b \\ -b^* & a^* \end{pmatrix} \tag{3}$$

$$\det U = aa^* + bb^* = |a|^2 + |b|^2 = 1 \tag{4}$$

Now, it is clear that the identity matrix I satisfies the conditions given in (3) and (4):

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in SU(2) \tag{5}$$

Let's see what the $SU(2)$ group looks like around the identity matrix. To this end, let

$$a = 1 + \delta\theta_a \tag{6}$$

$$b = 0 + \delta\theta_b \tag{7}$$

where a and b are defined in (3). From (4), we have:

$$\begin{aligned} (1 + \delta\theta_a)(1 + \delta\theta_a^*) + \delta\theta_b \delta\theta_b^* &= 1 \\ 1 + \delta\theta_a + \delta\theta_a^* + O(\delta\theta^2) &= 1 \end{aligned} \tag{8}$$

Since $O(\delta\theta^2)$ can be ignored when $\delta\theta$ is small, we conclude:

$$\delta\theta_a = -\delta\theta_a^* \tag{9}$$

which implies that $\delta\theta_a$ is purely imaginary. We also see that there is no restriction for $\delta\theta_b$. Therefore, we can write:

$$\delta\theta_a = \frac{i\delta\theta_3}{2} \tag{10}$$

$$\delta\theta_b = \frac{i\delta\theta_1}{2} + \frac{\delta\theta_2}{2} \tag{11}$$

for some real $\delta\theta_1$, $\delta\theta_2$, and $\delta\theta_3$. Here the factor 1/2 is included in the interest of convenience. Then, around the identity matrix, we can write U as:

$$\begin{aligned} U &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & \frac{i\delta\theta_1}{2} \\ \frac{i\delta\theta_1}{2} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \frac{\delta\theta_2}{2} \\ -\frac{\delta\theta_2}{2} & 0 \end{pmatrix} + \begin{pmatrix} \frac{i\delta\theta_3}{2} & 0 \\ 0 & -\frac{i\delta\theta_3}{2} \end{pmatrix} \\ &= 1 + i\frac{\sigma^1}{2}\delta\theta_1 + i\frac{\sigma^2}{2}\delta\theta_2 + i\frac{\sigma^3}{2}\delta\theta_3 \\ &= 1 + i\frac{\vec{\sigma}}{2} \cdot \delta\vec{\theta} \end{aligned} \tag{12}$$

where the σ's are the Pauli matrices. In fact, we can successively apply U's of this form around I to reach other elements of $SU(2)$; we can build up the infinitesimal generators into a finite one. In other words, any element of $SU(2)$ can be expressed in the following form for some finite $\vec{\theta}$:

$$\lim_{N\to\infty} \left(1 + i\frac{\vec{\sigma}}{2} \cdot \frac{\vec{\theta}}{N}\right)^N = e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}} \tag{13}$$

This equation can be checked by expanding the exponential in a Taylor series and comparing each power of $\vec{\theta}$ with the corresponding one on the lefthand side. In any case, the 2×2 matrix so obtained can be shown to satisfy (3) and (4)

Summarizing, we say the $SU(2)$ Lie group is "generated" by the Pauli matrices, which satisfy:

$$[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}] = i\epsilon^{ijk}\frac{\sigma^k}{2} \tag{14}$$

where we have used the Einstein-summation convention for the repeated index, and ϵ^{ijk} is the Levi-Civita symbol. Using slightly, different notation, we can reexpress the above equation as follows:

$$[T^i, T^j] = i\epsilon^{ijk}T^k \tag{15}$$

This defines the $SU(2)$ Lie algebra. For a general Lie algebra, we can write:

$$[T^i, T^j] = if^{ijk}T^k \tag{16}$$

where the f's are called the structure constants and the T's are called the generators.

In fact, it is known that in a Lie group all we need to know about the generators are their commutation relations. For example, one will never need to know how the products T^iT^j and T^jT^i can be expressed individually by other generators; the expression for their difference, namely $T^iT^j - T^jT^i$, suffices. This is so for the following reason. If we evaluate the product of two elements of a Lie group e^A, e^B as follows we get:

$$e^A e^B = e^{A+B+\delta} \tag{17}$$

where δ is obtained from the commutators of A and B as follows.

$$\delta = \frac{1}{2}[A, B] + \frac{1}{12}([A, [A, B]] + [[A, B], B]) + \frac{1}{24}[[A, [A, B]], B] + \cdots \tag{18}$$

This expression comes from the Baker-Campbell-Hausdorff theorem. Therefore, we see that a suitable basis for A and B is a set of T's whose commutators we know.

Finally, let us mention the Jacobi identity:

$$[T^i, [T^j, T^k]] + [T^j, [T^k, T^i]] + [T^k, [T^i, T^j]] = 0 \tag{19}$$

One can prove this by explicit calculations as follows:

$$\begin{aligned} &[T^i, T^jT^k - T^kT^j]] + [T^j, T^kT^i - T^iT^k]] + [T^k, T^iT^j - T^jT^i] = 0 \\ &T^i(T^jT^k - T^kT^j) - (T^jT^k - T^kT^j)T^i + \cdots = 0 \end{aligned} \tag{20}$$

The Jacobi identity gives a consistency condition for the structure constants, as follows:

$$\begin{aligned} &[T^i, if^{jkl}T^l] + [T^j, if^{kil}T^l] + [T^k, if^{ijl}T^l] = 0 \\ &f^{jkl}f^{ilm} + f^{kil}f^{jlm} + f^{ijl}f^{klm} = 0 \end{aligned} \tag{21}$$

Summary

- In the neighborhood of the identity matrix, an $SU(2)$ matrix is of the form

$$U = 1 + i\frac{\vec{\sigma}}{2} \cdot \delta\vec{\theta}$$

- More generally, an $SU(2)$ matrix is of the form

$$\lim_{N\to\infty} \left(1 + i\frac{\vec{\sigma}}{2} \cdot \frac{\vec{\theta}}{N}\right)^N = e^{i\vec{\theta}\cdot\frac{\vec{\sigma}}{2}}$$

- In such a case, we say $SU(2)$ Lie group is "generated" by the Pauli matrices which satisfy

$$[\frac{\sigma^i}{2}, \frac{\sigma^j}{2}] = i\epsilon^{ijk}\frac{\sigma^k}{2}$$

- More generally, a Lie Group is generated by Lie algebra, which satisfies

$$[T^i, T^j] = if^{ijk}T^k$$

  where the f's are called the structure constants and the T's are called the generators.

- Knowing such a commutator relation of Lie algebra is enough to generate a Lie group

- The Jacobi identity is given by

$$[T^i, [T^j, T^k]] + [T^j, [T^k, T^i]] + [T^k, [T^i, T^j]] = 0$$

  which gives relations between the structure constants.