Representations of the $SU(2)$ Lie algebra

In an earlier article, "$SU(2)$ Lie group and Lie algebra," we have seen that

$$[T^i, T^j] = i\epsilon^{ijk}T^k \tag{1}$$

defines the $SU(2)$ Lie algebra and that the 2×2 Pauli matrices "represent" the $SU(2)$ Lie algebra well. In other words, the Pauli matrices satisfy the above equations. Given this, a natural question arises: Are there any other representations of the $SU(2)$ algebra?

The answer is yes. Recall that the commutation relations for angular momentum in quantum mechanics follow the form given in (1). Pauli matrices are simply the components of angular momentum in a spin-1/2 system. Therefore, we can generalize Pauli matrices to a spin-1 system, a spin-3/2 system, and so on. These systems will give other representations of $SU(2)$. For example, for spin-1 system we have:

$$T_1 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \tag{2}$$

$$T_2 = \frac{1}{2\sqrt{2}} \begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \tag{3}$$

$$T_3 = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \tag{4}$$

These matrices satisfy (1). These matrices are called the spin-1 representation of $SU(2)$. Similarly, we have spin-3/2, 2, $5/2 \cdots$ representations. Notice that the spin-ℓ representation is given by matrices with dimensions $(2\ell + 1) \times (2\ell + 1)$.

Summary

- $L_x/2$, $L_y/2$, $L_z/2$ in various ℓ is actually the $(2\ell + 1)$-dimensional representation of $SU(2)$ Lie algebra.