The symmetric group $\mathcal{S}_n$ and the alternating group $\mathcal{A}_n$

In this article, we will introduce the "symmetric group" and "alternating group" as examples of group. The symmetric group $\mathcal{S}_n$ is a group of all permutations of a set of n elements. We encountered the concept of permutation in our earlier article "Bosons, Fermions, and Pauli's exclusion principle."

Let's consider why permutations form a group. We will consider the case when $n = 5$, as an example. Let's consider a permutation of $(1, 2, 3, 4, 5)$. If $(1, 2, 3, 4, 5)$ becomes $(2, 4, 3, 1, 5)$ after a permutation, we represent this permutation by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 1 & 5 \end{pmatrix} \tag{1}$$

In other words, in this permutation, we have

$$\begin{aligned} 1 &\to 2 \\ 2 &\to 4 \\ 3 &\to 3 \\ 4 &\to 1 \\ 5 &\to 5 \end{aligned} \tag{2}$$

The rule is that a number becomes whatever the number below the original number is in (1). Another common notation is following.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \sigma(5) \end{pmatrix} \tag{3}$$

In our case, we have $\sigma(1) = 2$, $\sigma(2) = 4$, $\sigma(3) = 3$ and so on.

Then, how many possibilities are there for a permutation of 5 objects? There are 5! possibilities: $\sigma(1)$ can take any value from 1 to 5, $\sigma(2)$ can take any value from 1 to 5 except for the one chosen for $\sigma(1)$, and $\sigma(3)$ can take any value from 1 to 5 except for the ones chosen for $\sigma(1)$ and $\sigma(2)$. The similar considerations can be made for $\sigma(4)$ and $\sigma(5)$. Thus, we indeed get $5 \times 4 \times 3 \times 2 \times 1 = 5!$; there are a total of 5! different functions for σ.

Regarding the expression of σ, there is actually no reason to write the first row in (1) in the alphabetical order. We could express the same permutation as (1) by

$$\sigma = \begin{pmatrix} 3 & 2 & 1 & 5 & 4 \\ 3 & 4 & 2 & 5 & 1 \end{pmatrix} \tag{4}$$

We still have $\sigma(3) = 3$, $\sigma(2) = 4$, $\sigma(1) = 2$, and so on just as before.

Now, we can think of the group operation for permutation. For example, consider

$$\sigma' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix} \tag{5}$$

In other words,

$$\begin{aligned} 1 &\to 3 \\ 2 &\to 1 \\ 3 &\to 5 \\ 4 &\to 2 \\ 5 &\to 4 \end{aligned} \tag{6}$$

Then, we can define $\sigma \bullet \sigma'$ by two successive permutations of σ' and σ. For example, from (2) and (6), $\sigma \bullet \sigma'$ is given by

$$\begin{aligned} 1 &\to 3 \to 3 \\ 2 &\to 1 \to 2 \\ 3 &\to 5 \to 5 \\ 4 &\to 2 \to 4 \\ 5 &\to 4 \to 1 \end{aligned} \tag{7}$$

In other words,

$$\sigma \bullet \sigma' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 5 & 4 & 1 \end{pmatrix} \tag{8}$$

or, more generally,

$$\sigma \bullet \sigma' = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \sigma(\sigma'(1)) & \sigma(\sigma'(2)) & \sigma(\sigma'(3)) & \sigma(\sigma'(4)) & \sigma(\sigma'(5)) \end{pmatrix} \tag{9}$$

So, we defined the group multiplication. Now, it is very easy to check that this group multiplication satisfies the four group axiom. First of all, it is obvious from our example that, if σ and σ' are two permutations, then $\sigma \bullet \sigma'$ is also a permutation. In other words, if σ and σ' are two elements of the symmetric group, $\sigma \bullet \sigma'$ is also an element of the symmetric group. Second, $\sigma \bullet (\sigma' \bullet \sigma'') = (\sigma \bullet \sigma') \bullet \sigma''$ is also obvious, as both expressions send

an element a to $\sigma(\sigma'(\sigma''(a)))$. Third, the identiy element of the symmetric group $\mathcal{S}_n$ is given by

$$e = \begin{pmatrix} 1 & 2 & \cdots & n-1 & n \\ 1 & 2 & \cdots & n-1 & n \end{pmatrix} \tag{10}$$

It is easy to check that $\sigma \bullet e = e \bullet \sigma = \sigma$. Fourth, the inverse element is also easy to find. For example, if σ is an element of $\mathcal{S}_5$, i.e.,

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & b & c & d & e \end{pmatrix} \tag{11}$$

we have

$$\sigma^{-1} = \begin{pmatrix} a & b & c & d & e \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \tag{12}$$

This completes the proof that the set of all permutations for n elements is indeed a group. This is a group called S_n.

How many elements are there in $\mathcal{S}_5$? Earlier, we have seen that there are 5! possible functions for σ. Therefore, there are 5! elements in S_5. In general, there are $n!$ elements in $\mathcal{S}_n$.

Problem 1. Explain why D_4, introduced in the last article, is a subgroup of $\mathcal{S}_4$.

Problem 2. Explain why $\mathcal{S}_4$ is a subgroup of $\mathcal{S}_5$.

Now, we will introduce the alternating group $\mathcal{A}_n$. First, recall from our earlier article "Bosons, Fermions, and Pauli's exclusion principle" what even permutation and odd permutation are. As before, we will give you examples when the number of permutated objects are 5. Let a_1, a_2, a_3, a_4, a_5 be grassmann numbers. Then, we had

$$\frac{a_{\sigma(i)}a_{\sigma(j)}a_{\sigma(k)}a_{\sigma(l)}a_{\sigma(m)}}{a_i a_j a_k a_l a_m} = \begin{cases} +1, & \text{if } \sigma \text{ is an even permutation} \\ -1, & \text{if } \sigma \text{ is an odd permutation} \end{cases} \tag{13}$$

In other words, a permutation which you can reach by even number of swaps is an even permutation, and a permutation which you can reach by odd number of swaps is an odd permutation. Every time you swap, you get a minus sign in the product of Grassmann numbers. If you swap even number of times, you get a positive sign, and if odd number of times, you get a negative sign.

Problem 3. Show that

- If σ is an even permutation, σ' is an even permutation, $\sigma \bullet \sigma'$ is an even permutation.
- If σ is an even permutation, σ' is an odd permutation, $\sigma \bullet \sigma'$ is an odd permutation.

- If σ is an odd permutation, σ' is an even permutation, $\sigma \bullet \sigma'$ is an odd permutation.
- If σ is an odd permutation, σ' is an odd permutation, $\sigma \bullet \sigma'$ is an even permutation.

Problem 4. Show that a group of even permutation forms a subgroup of $\mathcal{S}_n$. This group is called "alternating group" $\mathcal{A}_n$.

How many elements are there in an alternating group $\mathcal{A}_n$? We will now show that the number of even permutations and the number of odd permutations in $\mathcal{S}_n$ are the same, by matching each even permutation with each odd permutation. The one-to-one matching is given by

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \sigma(5) \end{pmatrix} \leftrightarrow \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \sigma(2) & \sigma(1) & \sigma(3) & \sigma(4) & \sigma(5) \end{pmatrix} \tag{14}$$

Notice that if the left one is an even permutation, the right one is an odd permutation, and vice versa. This is so, because

$$a_{\sigma(1)}a_{\sigma(2)}a_{\sigma(3)}a_{\sigma(4)}a_{\sigma(5)} = -a_{\sigma(2)}a_{\sigma(1)}a_{\sigma(3)}a_{\sigma(4)}a_{\sigma(5)} \tag{15}$$

As the numbers of even permutation and odd permutation are equal, the number of even permutation is half of the number of elements in $\mathcal{S}_n$. Thus, the alternating group $\mathcal{A}_n$ has $n!/2$ elements.

Final comment. In our earlier article "Quadratic equation," we brielfy mentioned that the French mathematician Galois proved that there is no general solution to quintic equations (i.e., $ax^5+bx^4+cx^3+dx^2+ex+f=0$) or higher-order equations. As it is impossible to explain his proof in a couple of pages, I just want to comment that he used the property of $\mathcal{S}_5$ to prove this. First, let me explain that the quintic equations are invariant under the action of $\mathcal{S}_5$ group in the solution. No. Let me just explain that the cubic equations ($ax^3+bx^2+cx+d=0$) are invarint under the action of $\mathcal{S}_3$ group, as you can figure out yourself the quintic case from the same reasoning. If the solution to a cubic equation is s_1, s_2, s_3, we have

$$ax^3+bx^2+cx+d = a(x-s_1)(x-s_2)(x-s_3) = 0 \tag{16}$$

which implies

$$b = -a(s_1+s_2+s_3), \qquad c = a(s_1s_2+s_1s_3+s_2s_3), \qquad d = -as_1s_2s_3 \tag{17}$$

Given this, notice that these equations (17) don't change under the permutations of s_1, s_2, s_3. For example, for the following permutation

$$\begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & s_3 & s_1 \end{pmatrix} \tag{18}$$

(16) becomes

$$ax^3 + bx^2 + cx + d = a(x - s_2)(x - s_3)(x - s_1) = 0 \tag{19}$$

which is exactly the same as (16), which implies (17) doesn't change either.

Summary

- A group of all permutations of a set of n elements is called the "symmetric group" S_n.
- The symmetric group S_n has $n!$ elements.
- A permutation is either an even permutation or an odd permutation.
- The even permutation in an symmetric group forms a group called "alternating group."
- The number of elements in an alterating group $\mathcal{A}_n$ is $n!/2$.