Symmetry and conservation law in quantum mechanics

In our earlier article "Noether's theorem," we have seen that there is a conserved charge for every symmetry. In this article, we will see how quantum picture of this looks like.

There, we have seen that if a Lagrangian is invariant under the deformation generated by Q, Q is conserved. i.e, $\frac{dQ}{dt} = 0$. This implies $\{Q, H\} = 0$ which in turn implies $[Q, H] = 0$. For example, if the x-component of momentum is conserved, $[P_x, H] = 0$ must be satisfied.

Problem 1. Consider a Hamiltonian given by

$$H = \frac{P_x^2}{2m} + V(x) \tag{1}$$

Then, show that the x-component of momentum is conserved, only when $V(x)$ is constant. (Of course, this is obvious as there is no force, only when the potential is constant. We can also see this from Noether's theorem picture. If the potential is constant, the system has a spatial translational symmetry as $V(x) = V(x + \epsilon)$.)

Now, we will see how a particular quantity q changes under Q. From "Noether's theorem" article, we know $\epsilon\{q, Q\} = \delta q$. In other words,

$$\epsilon\{q(\alpha), Q\} = q(\alpha + \epsilon) - q(\alpha) \tag{2}$$

Let's find its quantum analogue. As an example, we will consider the space translation by x direction for the generator Q. For infinitesimal distance ϵ, we have

$$\begin{aligned} \psi(x + \epsilon) &= \psi(x) + \epsilon\frac{\partial\psi(x)}{\partial x} &(3)\\ &= \psi(x) + \frac{iP_x}{\hbar}\epsilon\psi(x) &(4)\end{aligned}$$

Actually, we have already seen this when you proved

$$e^{iP_x a/\hbar}\psi(x) = \psi(x + a) \tag{5}$$

in our earlier article "A short introduction to quantum mechanics X."

The expectation value of $q(x)$ is given by

$$\langle q(x)\rangle = \int \psi^*(x)q(x)\psi(x)dx \tag{6}$$

Similarly,

$$\begin{aligned}
\langle q(x+\epsilon)\rangle &= \langle\psi|q(x+\epsilon)|\psi\rangle = \int \psi^*(x)q(x+\epsilon)\psi(x)dx && (7)\\
&= \int \psi^*(x-\epsilon)q(x)\psi(x-\epsilon)dx && (8)\\
&\approx \int \psi^*(x)\left[q(x)+\frac{iP_x}{\hbar}\epsilon q - q\frac{iP_x}{\hbar}\epsilon\right]\psi(x)dx && (9)
\end{aligned}$$

Thus,

$$\langle\delta q\rangle = \langle q(x+\epsilon)-q(x)\rangle = \frac{\epsilon}{i\hbar}\langle[q,P_x]\rangle \tag{10}$$

Notice that this relation is exactly (2), with $Q = P_x$, and the Poisson bracket appropriately replaced by the commutator! Here, we only considered the space translation, one can easily derive other cases such as time translation and rotation similarly.

Now, remember that $e^{iP_x a/\hbar}$ in (5) generates the translation of finite (i.e. not infinitesimal) distance a. Similarly, if Q generates a symmetry of a physical system, the unitary operator that shifts the coordinate α to $\alpha + a$ under such a symmetry is given by $U(a) = \exp(iQa/\hbar)$. In other words,

$$U(a)\psi(\alpha) = \psi(\alpha + a) \tag{11}$$

Of course, now α can mean other coordinates than positions such as time or angle. If Q is conserved, from $[Q,H] = 0$, we can derive

$$[U(a),H] = 0 \tag{12}$$

Given this, consider an eigenstate of Hamiltonian $|\psi\rangle$ with eigenvalue E. We will assume that the eigenvalues are all distinct; there is at most one eigenstate with a given eigenvalue. Then, from (12), we have

$$\begin{aligned}
HU(a)|\psi\rangle &= U(a)H|\psi\rangle && (13)\\
HU(a)|\psi\rangle &= U(a)E|\psi\rangle && (14)\\
H(U(a)|\psi\rangle) &= E(U(a)|\psi\rangle) && (15)
\end{aligned}$$

Thus we see that $U(a)|\psi\rangle$ is also an eigenstate of Hamiltonian with eigenvalue E. As there is at most one eigenstate with a given eigenvalue, $U(a)|\psi\rangle$ must be proportional to $|\psi\rangle$. As $U(a)$ is a unitary operator, we must have

$$U(a)|\psi\rangle = e^{i\phi(a)}|\psi\rangle \tag{16}$$

Now, remember (11). Thus, we have

$$U(a)\psi(\alpha) = e^{i\phi(a)}\psi(\alpha) = \psi(\alpha + a) \tag{17}$$

In conclusion, we have

$$e^{i\phi(a)}\psi(\alpha) = \psi(\alpha + a) \tag{18}$$

for some real $\phi(a)$ i.e., some phase $e^{i\phi(a)}$.

On the other hand, we could have arrived at the same conclusion approaching it from a different direction. If the system has the symmetry under $\alpha \to \alpha + a$, we might want to need

$$\psi(\alpha) = \psi(\alpha + a) \tag{19}$$

However, according to the concept of global gauge transformation, $\psi(\alpha)$ and $e^{i\theta(a)}\psi(\alpha)$ describe the same physics. Thus, we can relax the above condition to

$$e^{i\theta(a)}\psi(\alpha) = \psi(\alpha + a) \tag{20}$$

which is exactly (18) upon the identification $\theta(a) = \phi(a)$!

Problem 2. Show that $\theta(a)$ must be proportional to a. (Hint[1])

In this article, we have considered only continuous symmetries. Symmetry of type $\alpha \to \alpha + a$, where a can be any real number is called continuous symmetry. In the next article, we will consider a discrete symmetry called "parity."

Summary

- Noether's theorem can be equally applied to quantum mechanics.
- Symmetry of type $\alpha \to \alpha + a$, where a can be any real number is called continuous symmetry.

[1] Show $\theta(a) + \theta(b) = \theta(a + b)$.