The definition of temperature

Let's say that system 1 and system 2 are in thermal contact. This means that they can exchange energy. Let's also assume that the volume of each system is fixed, and that they cannot exchange actual material, but only energy.

Given this, remember our earlier formula:

$$S = k \ln W \tag{1}$$

where S is the entropy and W is the number of microstates consistent with a given macrostate. Now, observe that if the W for system 1 is W_1 and the W for system 2 is W_2, the total W is $W_1 W_2$. If you don't understand why it should be a product rather than a sum, let me explain with an example. If you cast a die and a coin so that you get 6 outcomes for the die and 2 outcomes for the coin (i.e. head or tail), there are $12 = 6 \times 2$ possible outcomes in total. Using this relation, the total entropy is given by:

$$S = k \ln W = k \ln(W_1 W_2) = k \ln W_1 + k \ln W_2 = S_1 + S_2 \tag{2}$$

where S_1 is the entropy of system 1 and S_2 the entropy of system 2. You see that the total entropy is just the sum of each component.

Now, let's denote the energy of system 1 by E_1 and the energy of system 2 by E_2. Given this, let's say that energy E is transferred from system 1 to system 2 during a thermal contact. This happens only when T_1, the temperature of system 1, is higher than T_2, the temperature of system 2, since spontaneously, energy always flows from a hot place to a cold place. What is the total change in entropy? The energy of system 1 has changed by $-E$ while the energy of system 2 has changed by $+E$. Therefore,

$$\Delta S = \Delta S_1 + \Delta S_2 = \frac{\partial S_1}{\partial E_1}\Delta E_1 + \frac{\partial S_2}{\partial E_2}\Delta E_2 = (-\frac{\partial S_1}{\partial E_1} + \frac{\partial S_2}{\partial E_2})E \tag{3}$$

Now, we know that entropy always increases. This implies:

$$\frac{\partial S_1}{\partial E_1} < \frac{\partial S_2}{\partial E_2} \tag{4}$$

Remembering that $T_1 > T_2$, the above inequality all makes sense, if

$$\frac{\partial S_1}{\partial E_1} = \frac{1}{T_1}, \qquad \frac{\partial S_2}{\partial E_2} = \frac{1}{T_2} \tag{5}$$

Turning it the other way around, we can actually say that the above equations *define* temperature. Notice also that the two systems will be in thermal equilibrium if the entropy is maximized, after a lot of energy exchange. In other words,

$$0 = \Delta S = (-\frac{\partial S_1}{\partial E_1} + \frac{\partial S_2}{\partial E_2})E \tag{6}$$

which implies:

$$\frac{\partial S_1}{\partial E_1} = \frac{\partial S_2}{\partial E_2} \tag{7}$$

which in turn implies $T_1 = T_2$. Therefore, if two systems are in thermal equilibrium, their temperatures must be the same.

A final remark. In this article, we have seen that the definition of temperature is given by:

$$\frac{1}{T} = \frac{\partial S}{\partial E} \tag{8}$$

In other words,

$$\frac{1}{T} = \frac{\Delta S}{\Delta E}, \qquad \text{or} \qquad \Delta S = \frac{\Delta E}{T} \tag{9}$$

So, we obtain our earlier formula $\Delta S = \Delta Q/T$. In this article, we called E what we called Q in our earlier article.

Problem 1. Explain why an object with a negative temperature is hotter than any object with a positive temperature. (Hint[1])

Summary

- Temperature is defined by $\dfrac{1}{T} = \dfrac{\partial S}{\partial E}$.
- That energy always flows in such a direction that the total entropy increases implies that energy always flows from a hot place to a cold place.

[1] Show that the energy will flow from the former to the latter for the entropy to increase. You will need to use (8).