The tensor

Tensors are important objects in physics. It is a further generalization of scalars, vectors, and matrices. We will use tensors extensively in our later articles on general relativity.

A rank (p, q) tensor is an object with p upper indices and q lower indices. For example, T_{ij}^{klm} is a (3,2) tensor. A scalar has no indices. Therefore, it's a rank $(0, 0)$ tensor. A vector v^i has one upper index. Therefore, it's a rank $(1, 0)$ tensor. A covector u_j has one lower index. Therefore it's a rank $(0, 1)$ tensor.

A rank (p, q) tensor is also called a rank $(p + q)$ tensor, if the number of upper and lower indices is unimportant. It is easy to see that a matrix is a rank 2 tensor, because it has two indices.

Notice that the elements of a matrix can be naturally listed in a two-dimensional representation, because it has two indices; for example,

$$a_{ij} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \tag{1}$$

Here, you see that the two indices i and j parametrize the two-dimensional representation. Also, as this example illustrates, an $n \times n$ matrix has n^2 elements. Notice that the exponent 2 comes from the fact that a matrix is a rank 2 tensor.

Similarly, a vector, which is a rank 1 tensor can be naturally listed in a linear one-dimensional representation; for example,

$$v^i = \begin{pmatrix} v^1 & v^2 & v^3 \end{pmatrix} \tag{2}$$

As you see here, a vector (a rank 1 tensor) in n dimensional space has n^1 elements.

Similarly, a scalar has only one element; in other words, it has n^0 elements, because it is a rank-0 tensor. Now, you see that you can regard rank 3 tensor as an $n \times n \times n$ "matrix," except that it is not a matrix. It has n^3 elements, and although there is no natural way to represent its elements in two dimensions, they can be listed in a cube.

When you deal with general relativity or quantum field theory, you will perform many tensor calculations. The calculation will necessarily involve the Einstein summation convention. A good example of tensor calculation which you are already familiar with would be following:

$$\delta^a_b A^b_c = A^a_c \tag{3}$$

I first didn't intend to write this part of the article, but I found out that some students made silly mistakes in tensor calculations, so let me mention some grammatical rules in tensor calculations, even though they seem obvious, once you get used to it.

First of all, the free indices must match on both sides. For example,

$$A^a = B^a + C^{ab} D_b \tag{4}$$

is correct, but

$$E^a = F^b + G^b \tag{5}$$

is not correct. It has to be either $E^a = F^a + G^a$ or $E^b = F^b + G^b$. Similarly,

$$G^a_{bc} = H_{abc} \tag{6}$$

is not correct. If the left-hand side if a rank (1,2) tensor, the right-hand side must be also a rank (1,2) tensor, not a rank (0,3) tensor.

Second, the Einstein summation convention means that a dummy variable must be used as an upper index and as a lower index only once each time. For example,

$$J_a = K^b_a L_b M_b \tag{7}$$

is wrong, because b is used as a lower index twice. Similarly,

$$N_a = P^b_a Q_b R^b_b \tag{8}$$

is also wrong. However,

$$S_a = T^b_a (U_b + V_b) \tag{9}$$

is correct, as we can see $U_b + V_b$ as a single rank (0,1) vector with index b or, we could regard the above equation as

$$S_a = T^b_a U_b + T^b_a V_b \tag{10}$$

which is also fine.

Also, our second grammatical rule means that index cannot be used both as a free index and a dummy index. For example,

$$W_a = X_{ab} Y^{ba} Z_a \tag{11}$$

is wrong.

Summary

- A rank (p, q) tensor has p upper indices and q lower indices.
- A rank (p, q) tensor is a rank $(p + q)$ tensor.
- A scalar is a rank (0,0) tensor, a vector is a rank (1,0) tensor, a covector is a (0,1) tensor, and a matrix is a rank 2 tensor.