Time dilation in Einstein's theory of special relativity

In this article, from the constancy of the speed of light, we will derive the fact that a moving clock ticks at a slower rate than a non-moving one.

Consider a clock that consists of the two mirrors A and B. It sends a light pulse at mirror B and it bounces at mirror A and comes back to mirror B. As the distance between two mirrors is L, in the frame that the clock is not moving, the whole trip will take the following amount of time:

$$\Delta t = \frac{2L}{c} \tag{1}$$

where c is the speed of light. Now, suppose you are moving leftward with speed v, then you will see that the clock moves rightward with speed v. See Fig.2. Then, the same trip considered before will take longer distance in this case. It has to move $2D$ which is bigger than $2L$. As the speed of light is constant, you will see that the whole trip takes the following amount of time:

$$\Delta t' = \frac{2D}{c} \tag{2}$$

which is bigger than Δt, as D is bigger than L. Therefore, you see that $\Delta t'$ amount of time passes while only Δt amount of time passes for moving clock. You see that the clock moving goes at a slower rate. Notice that the constancy of the speed of light plays an important role here. If there were no such a rule, the denominator in (2), the speed of light in the second case, would be bigger than the denominator in (1), the speed of light in the first case. This

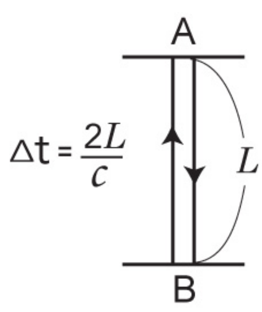

Figure 1: clock at rest

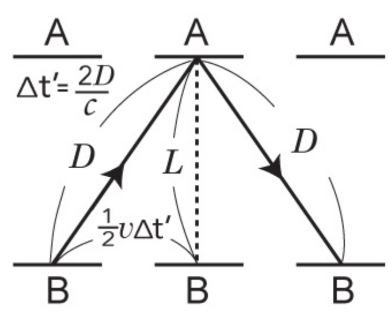

Figure 2: moving clock

would imply that $\Delta t'$ would be same as Δt. Precisely speaking, the speed of light in the second case would be not c, but $\sqrt{c^2+v^2}$.

One can actually calculate at which rate the clock goes slower. See Fig.2 again. As the clock moves $\frac{1}{2}v\Delta t'$ amount of distance rightward when the light pulse makes half the trip, we have, by using Pythagorean theorem:

$$D = \sqrt{(\frac{1}{2}v\Delta t')^2 + L^2} \tag{3}$$

Applying (2) and (1) to this equation, we obtain:

$$\Delta t' = \frac{2L/c}{\sqrt{1-v^2/c^2}} = \frac{\Delta t}{\sqrt{1-v^2/c^2}} \tag{4}$$

As the denominator is less than 1, we see that $\Delta t'$ is bigger than Δt.

Problem 1. Suppose John travels on a commercial airplane with a constant speed 250 m/s at a constant height 8.8 kilometers, and Kerry is on Mountain Everest at the same height 8.8 kilometers. How many hours have elapsed for Kerry during the time in which John's watch elapses 10 hours? (250 m/s and 8.8 km are the common speed and the common height for a commercial airplane. We assume that John and Kerry are at the same height, as the rate the time ticks depends not only on the speed but also on the height.) (Hint[1]) The answer is 10 hours and 1.25×10^{-8} seconds. So, the time dilation effect is not quite noticeable in our daily lives. This is so from the following reason. In our daily lives v (in our case 250 m/s) is much smaller than c (i.e. 300,000 km/s), which makes the difference between $\Delta t'$ and Δt very small, as $\sqrt{1-v^2/c^2}$, the denominator of (4) is almost equal to 1 in such a case. Nevertheless, as we will mention in our later article "Twin paradox," the time dilation effect is confirmed by atomic clocks and commercial airplane in 1971.

Problem 2. A space-ship travels from the earth to a distant star 10 light-years away at a constant velocity. (1 light year is the distance that light travels in a year.) If an astronaut in the space-ship measures that his trip takes 10 years, what is the speed of the space-ship? (Hint[2])

Summary

- $\Delta t' = \dfrac{\Delta t}{\sqrt{1-v^2/c^2}}$. Moving clock ticks at a slower rate than a non-moving one.

[1]The result in Problem 1. of our earlier article "The imagination in mathematics: "Pascal's triangle, combination, and the Taylor series for square root" will be helpful as the usual calculator is not normally sensitive enough to accurately calculate the square root of a number that is so close to 1 such as the one in this case.

[2]The time the trip took is longer than 10 years from the point of view of an observer at the earth because of time dilation. Also, the time the trip took from the point of view of the observer at the earth is given by 10 light years divided by the speed of the space-ship. Assume that the speed of the space-ship is v and try to write these two conditions in terms of v, and equate them.