Topology, the Euler characteristic, and the Gauss-Bonnet theorem

1 Topology

At the end of our essay "Is math and science homework mechanical drudgery?" we introduced the concept of "topology." Topology is now a huge subject, and one of the major branches in mathematics, where many mathematicians work. In our example of topology in that article, we dealt with lines on a two-dimensional plane, but we could deal with the surfaces in 3-dimensional space. For example, a sphere and an spheroid (i.e., squeezed sphere) have the same topology. See Fig. 1 for a spheroid. You can turn a sphere into an spheroid without cutting or gluing. All you need to do is squeezing. Similarly, a torus, and a cup with one handle has the same topology, as you can change the former into the latter without cutting or gluing. See Fig. 2. That is the reason why there is a joke that topologists (i.e., mathematicians who study topology) cannot distinguish between a coffee cup and a donut.

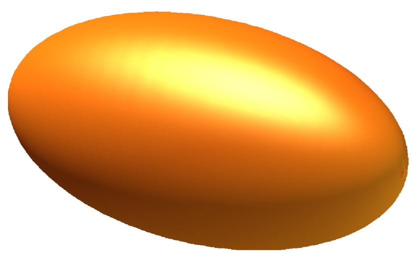

Figure 1: 3D representation of a spheroid. [1]

Figure 2: A coffee mug changing into a donut without topology change. [2]

However, you cannot turn a sphere into a torus without cutting or gluing. Therefore, a sphere and a torus have the different topology. If you are not sure what I mean, see Fig. 3. I turn a sphere into a spheroid then to a torus by gluing. There was a topology change between the third figure and the fourth figure. You see that two separate points A and B became the same point, which means that the connectedness has changed.

Of course, there are other two-dimensional surfaces which are neither sphere nor torus. See Fig. 4. and Fig. 5. They are topologically distinct from sphere and torus. Fig. 4 is called “2-holed torus” and Fig. 5 “3-holed torus.”

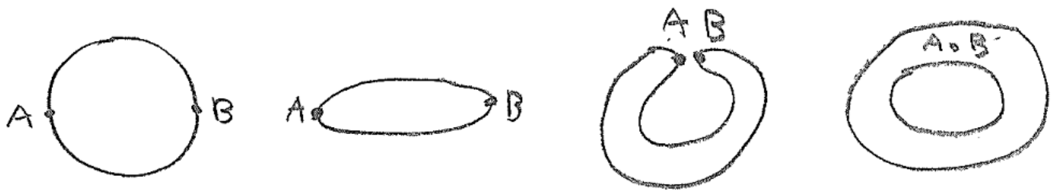

Figure 3: Planar projection of a sphere turning into a torus by a topology change

Figure 4: 2-holed torus. [3]

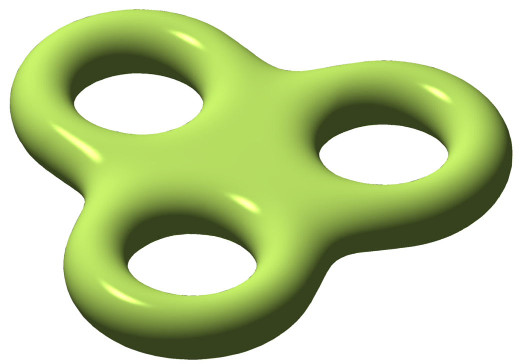

Figure 5: 3-holed torus. [4]

Before moving onto the topological classification of two-dimensional surfaces, let me introduce an important concept. There are two kinds of 2-dimensional surfaces. Those with boundaries, and those without boundaries. See Fig. 6 for a disk that has one boundary, and see Fig. 7 for a cylinder that has two boundaries.

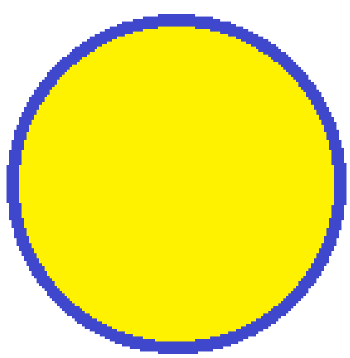

Figure 6: Representation of a disk

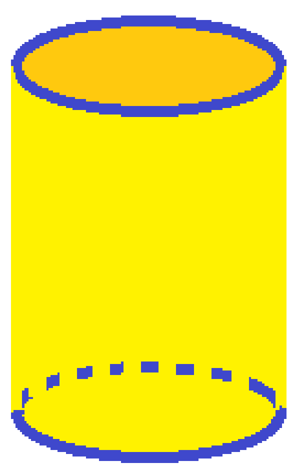

Figure 7: Representation of a cylinder

The boundaries are denoted by blue lines. As in our earlier article on manifold, mathematicians denote only the side of cylinder when they call something "cylinder." The two flat disks at the bottom and the top are not part of a cylinder. Otherwise, a cylinder would have the same topology as a sphere, having no boundary. Unlike a disk or a cylinder, we see that sphere, torus, 2-holed torus, and 3-holed torus don't have boundaries. For 2-dimensional surfaces, which have no boundaries, "genus" is defined by the number of holes. For example, the genus of sphere is 0, the genus of torus is 1, and the genus of g-holed torus is g. We see that two objects have different topology if their genuses are different.[1]

Before going on to the next section, let me mention that a disk is not the only 2-dimensional surface that has exactly one boundary. There is another topologically distinct surface that has exactly one boundary. Can you imagine? German mathematician August Möbius found one in the mid 19th century. More on this in our next article.

2 The Euler characteristic

In this section, we will talk about the Euler characteristic, and see how it is related to topology. There are three-dimensional objects called "Platonic solids." A Platonic solid has the congruent regular (all sides and angles equal) polygons as its faces, and the same number of edges meet at each vertex. They are "3d" versions of regular polygons. Each vertex in a regular polygon is equivalent when it relates to the positions of other vertices. The same can be said about the Platonic solids. There are only five Platonic solids. See Fig. 8.

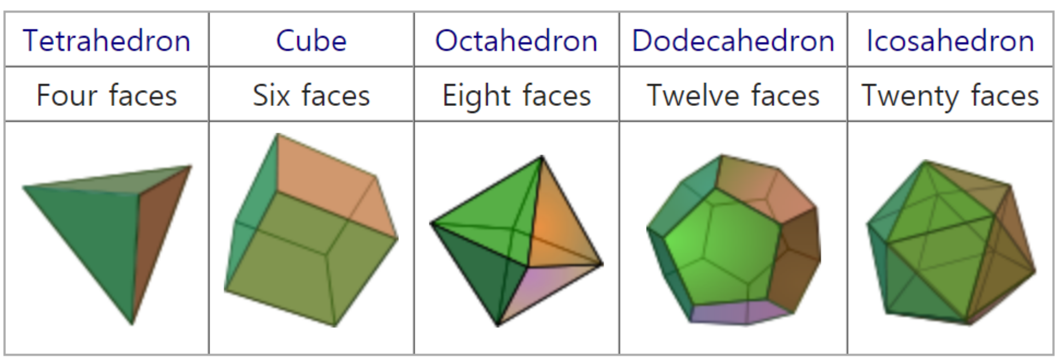

Figure 8: Graphic representation of Platonic solids. These 3D shapes have the property of being regular, which means all sides and angles are equal. [5].

In Table 1, you see number of vertices, edges and faces of each Platonic solid. Notice that $v + f = e + 2$ is satisfied by all the Platonic solids. Is this a coincidence? Actually, note that all the Platonic solids have genus zero. So, one may suspect that this may not be a property of the Platonic solids, but a property of an object with genus zero.

[1]Genus can be also defined for the surfaces with boundaries, but the definition is beyond scope of this article.

Platonic solid	vertices (v)	edges (e)	faces (f)
Tetrahedron	4	6	4
Cube	8	12	6
Octahedron	6	12	8
Dodecahedron	20	30	12
Icosahedron	12	30	20

Table 1: Number of vertices, edges and faces of Platonic solids.

Let's check it. In Fig. 9, you see some examples of prism and in Table 2, you see the number of their vertices, edges and faces. You see that $v + f = e + 2$ is satisfied again. Let's also check for cones. In Fig. 10, you see some examples of cone and in Table 3, you see the number of their vertices, edges and faces.

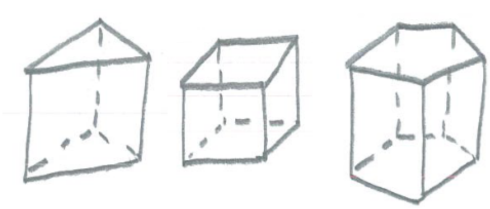

Figure 9: a triangular prism, a rectangular prism, and a pentagonal prism

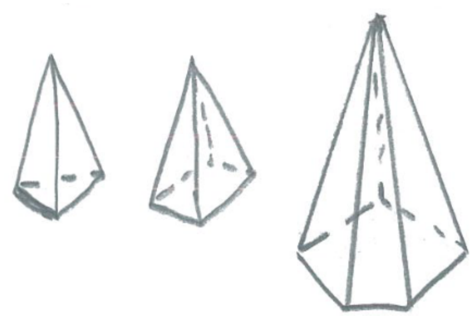

Figure 10: a triangular cone, a rectangular cone, and a pentagonal cone

Prism type	Vertices	Edges	Faces
Triangular	6	9	5
Rectangular	8	12	6
Pentagonal	10	15	7
n-polygonal	$2n$	$3n$	n+2

Table 2: The number of vertices, edges and faces of prisms

Cone type	Vertices	Edges	Faces
Triangular	4	6	4
Rectangular	5	8	5
Pentagonal	6	10	6
n-polygonal	$n + 1$	$2n$	$n + 1$

Table 3: The number of vertices, edges and faces of cones

You see that $v + f = e + 2$ is satisfied both for prisms and cones which have genus zero just like the Platonic solids. Thus, we can *suspect* that $v + f = e + 2$ is always satisfied for genus zero, 2-dimension surfaces without boundary, although we cannot be sure. So, let's check whether $v + f = e + 2$ is not satisfied for an object whose genus is not zero. See Fig. 11 for an example of object with genus 1. You see that $v = 16$, $e = 32$, $f = 16$. Thus, we see that $v + f = e$ is satisfied satisfied instead of $v + f = e + 2$.

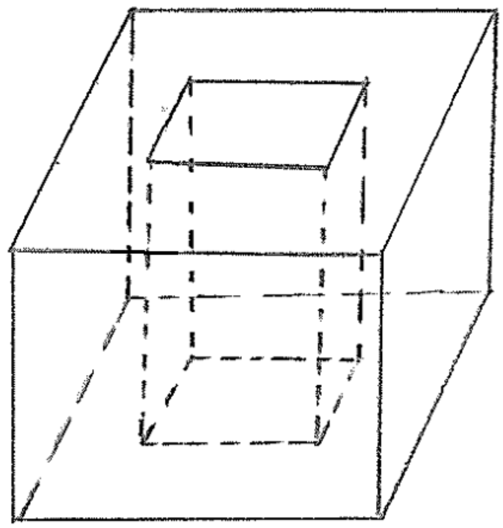

Figure 11: an example of genus-1 object

Let's summarize. We have emperically found

$$v - e + f = 2 \quad \text{(for genus zero)} \tag{1}$$

We *suspect* that this formula is correct, but we cannot be sure, because we have not proved it. Who knows if someone comes up with a counter-example? Luckily, Swiss mathematician Leonhard Euler, who solved the Königsberg problem, also proved (1) in 1758. We will not present the proof, but if you are interested you can read [6] where you can find twenty different proofs.

Now, let's define the Euler characteristic. The Euler characteristic χ is defined by

$$\chi \equiv v - e + f \tag{2}$$

For example, a sphere, which has a genus zero, has the Euler characteristic 2, and the object in Fig. 11 has the Euler characteristic 0.

It is also proved that the following relation between the Euler characteristic χ and genus g is satisfied for orientable, closed (i.e., without boundary) surfaces. (We will explain what is orientable in the next article.)

$$\chi = 2 - 2g \tag{3}$$

For example, a sphere has the Euler characteristic 2, because it has genus zero. $(2 = 2 - 2 \cdot 0)$. A torus, as well as the object in Fig. 11, has the Euler characteristic 0, because it has genus 1. $(0 = 2 - 2 \cdot 1)$

Problem 1. What is the Euler characteristic of two-holed torus?

3 Gauss-Bonnet theorem

In our earlier article "Curved space," we introduced the Gaussian curvature. Recall that the Gaussian curvature of a sphere with radius R is given by $1/R^2$. Recall also that the surface area of a sphere is given by $4\pi R^2$. Then, if we define the "total Gaussian curvature" by the Gaussian curvature multiplied by area, we see that the total Gaussian curvature of a sphere is given by 4π. (The "total Gaussian curvature" is my own terminology. Mathematicians don't use this terminology. I just made it up to explain things more easily.) Notice that the total Gaussian curvature of a sphere does not depend on the radius.

How can we calculate the total Gaussian curvature of an object which has non-constant Gaussian curvatures? We can divide its surface into small segments and multiply the Gaussian curvature of each segment by its area and sum them like this:

$$\sum_i K_i \times A_i = \text{the total Gaussian curvature} \tag{4}$$

where i denotes the ith segment, K_i, the Gaussian curvature of ith segment, A_i, the area of ith segment. Just like the cases in the last article, in which the volume and the area can be calculated more accurately as you slice them into smaller segments, the total Gaussian curvature can be calculated more accurately if you slice them into smaller segments. Of course, in the case of constant Gaussian curvature (i.e., sphere), the calculation is already accurate, because you get the exact result with few slices as follows:

$$\sum_i \frac{1}{R^2} A_i = \frac{1}{R^2} \sum_i A_i = \frac{1}{R^2} 4\pi R^2 = 4\pi \tag{5}$$

where I pulled out the factor $1/R^2$ out of the summation because it is a common factor that doesn't depend on slices to slices (i.e., the index i).[2]

Just in case you already know calculus, (4) can be expressed as

$$\int K\, dA = \text{the total Gaussian curvature} \tag{6}$$

If you want to know 100% sure what this expression means, you can come back to it after you learn calculus.

Anyhow, what would be the total Gaussian curvature of spheroid? See Fig. 12. The regions around "North Pole" and "South Pole" are relatively flat (i.e., small Gaussian curvature), but have large areas. On the other hand, the region around "equator" is quite curved (i.e., big Gaussian curvature), but has a small area. When you take into account everything, they compensate exactly each other and the total Gaussian curvature of spheroid is *exactly* 4π, just like the one of sphere. Of course, I can not demonstrate to you that it is exactly 4π, because I haven't taught you calculus yet, but anyhow mathematicians found that it is 4π.

Then, what would be the total Gaussian curvature of torus? If you recall the figure of torus in our earlier article "Curved space," you see that roughly half of its surface has positive

[2] If you are not sure what I mean, consider this example. $2 \times 1 + 2 \times 3 + 2 \times 2 + 2 \times 5 = 2(1 + 3 + 2 + 5)$

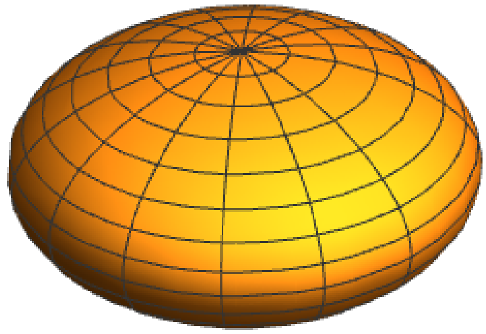

Figure 12: the region around “North Pole” and “South Pole” of this spheroid is relatively flat, while the region around its “equator” is relatively curved.

Gaussian curvature and and roughly half of its surface has negative Gaussian curvature. If you calculate the total Gaussian curvature accurately, it is *exactly* zero. Again, I cannot demonstrate to you, but mathematicians proved that it is zero.

Now, I will state the Gauss-Bonnet theorem. The Gauss-Bonnet theorem says that the total Gaussian curvature is given by $2\pi\chi$ where χ is the Euler characteristic. For example, the Euler characteristic of a sphere or an spheroid is 2, so the total Gaussian curvature is 4π. Similarly, the Euler characteristic of torus is zero, so the total Gaussian curvature is also zero.

The Gauss-Bonnet theorem was discovered in the 19th century. What is actually amazing is that mathematicians found its generalizations in the 20th century. The Gauss-Bonnet theorem only applies to 2-dimensional surface. The Chern theorem applies to $2n$-dimensional space, where n is a positive integer. The Gauss-Bonnet theorm is a special case of the Chern theorem when $n = 1$. The Chern theorem has wide applications in condensed matter physics and string theory. For example, the Nobel Prize in Physics 2016 was awarded to three condensed matter physicists “for theoretical discoveries of topological phase transitions and topological phases of matter.” They used the Chern theorem in their work.

Final comment. In this article, we presented two ways to calculate the Euler characteristic: by counting the number of vertices, edges and faces, and by calculating the “total Gaussian curvature.” In other words, we learned that Gaussian curvature was related to topology. This is somewhat unexpected, because topology doesn’t really care much about how curved each point on a surface is, but only cares about its overall shape. Nevertheless, when you add up the Gaussian curvature on whole area, you get a value related to topology. In other words, differential geometry is related to topology. Actually, there is a branch in topology called ”differential topology.” You can learn more about the Gauss-Bonnet theorem from a differential topology textbook. In our later article “The duality between de Rham cohomology and homology," we will present yet another way to calculate the Euler characteristic. A branch of topology called "algebraic topology" closely treats such topics. It is interesting that different areas in mathematics are unexpectedly related to one another.

Summary

- If you can change an object into another object without gluing or cutting (i.e., without changing their connectedness), the two objects have the same topology. Similarly, if you cannot change an object into another object without gluing or cutting (i.e., without changing their connectedness), the two objects have different topology.
- For example, a sphere and a spheroid have the same topology, and a coffee cup and a torus have the same topology. On the other hand, a sphere and a torus have different topology.
- For orientable, closed, two-dimensional surfaces, genus denotes the number of holes. For example, a sphere has genus 0, a torus genus 1, a 2-holed torus genus 2.
- If two surfaces have different genus, they have different topology.
- The Euler characteristic is defined by

  $$\chi \equiv v - e + f$$

  where v is the number of vertices, e the number of edges, and f the number of faces.
- For orientable, closed, two-dimensional surface, the Euler characteristic satisfies

  $$\chi = 2 - 2g$$

  where g is the genus of the surface.
- The Gauss-Bonnet theorem relates the "total" Gaussian curvature of a 2-dimensional surface with its Euler characteristic.

References

[1] `https://commons.wikimedia.org/wiki/File:Spheroid.png`

[2] Reproduced with permission of Prof. Henry Segerman. `https://www.youtube.com/watch?v=9NlqYr6-TpA`

[3] `https://commons.wikimedia.org/wiki/File:Double_torus_illustration.png`

[4] `https://commons.wikimedia.org/wiki/File:Triple_torus_illustration.png`

[5] https://en.wikipedia.org/wiki/Platonic_solid https://commons.wikimedia.org/wiki/File:Tetrahedron.svg https://commons.wikimedia.org/wiki/File:Hexahedron.svg https://commons.wikimedia.org/wiki/File:Octahedron.svg https://commons.wikimedia.org/wiki/File:Dodecahedron.svg https://commons.wikimedia.org/wiki/File:Icosahedron.svg

[6] Twenty proofs of Euler's formula V-E+F=2 https://www.ics.uci.edu/~eppstein/junkyard/euler/