The trace

Trace is defined as the total sum of diagonal elements of a square matrix. For example, if we have a following matrix:

$$A = \begin{pmatrix} -2 & 4 & -1 & 0 \\ -1 & 3 & -1 & 5 \\ 2 & -2 & 1 & -2 \\ 0 & 0 & 1 & 4 \end{pmatrix} \tag{1}$$

its trace is given by following:

$$\mathrm{tr}(A) = -2 + 3 + 1 + 4 = 6 \tag{2}$$

In other words, for an $n \times n$ matrix A_{ij}, the trace is given by:

$$\mathrm{tr}(A) = \sum_{i=1}^{n} A_{ii} \tag{3}$$

Using this notation, it is easy to check the following:

$$\mathrm{tr}(AB) = \mathrm{tr}(BA) \tag{4}$$

since

$$(AB)_{ij} = \sum_k A_{ik} B_{kj} \tag{5}$$

$$\begin{aligned} \mathrm{tr}(AB) &= \sum_i (AB)_{ii} && (6) \\ &= \sum_i \sum_k A_{ik} B_{ki} = \sum_k \sum_i B_{ki} A_{ik} && (7) \\ &= \sum_k (BA)_{kk} = \mathrm{tr}(BA) && (8) \end{aligned}$$

Problem 1. Prove the following:

$$\mathrm{tr}(ABC) = \mathrm{tr}(BCA) = \mathrm{tr}(CAB) \tag{9}$$

Comment: More generally, we have

$$\mathrm{tr}(ABC \cdots Z) = \mathrm{tr}(BC \cdots ZA) = \mathrm{tr}(CD \cdots ZAB) = \cdots = \mathrm{tr}(ZABC \cdots Y)$$

Summary

- Trace is the sum of diagonal elements of a square matrix.
- $\mathrm{tr}(AB) = \mathrm{tr}(BA)$