Travelling spring waves

In the earlier article, we considered the case in which two or three objects were connected to the springs or pendulums. In this article, we will consider the case that involves many objects, but each has the same mass and is connected to two springs with equal, universal spring constant. At first glance, the task at our hand may seem overwhelming; if there are n-objects, the equation for ω^2 that one can obtain from the determinant is an n-th order equation which is very hard to solve. Nevertheless, as we will see, the fact that all the objects and all the springs are same, simplifies the problem greatly, and allows us to dispense with calculating determinants.

See Fig.1. We have infinitely many objects with mass M and springs with spring constant K. They are oscillating, and we named the objects by integer n. In the figure, a particular mode is depicted. At the moment, the objects are packed around the object 3 (i.e. dense) and the objects are least packed around the object 0 and object 6 (i.e. light). It is easy to imagine that this pattern will go on as n increases. For example, objects near objects 9, 15, 21 will be dense and objects near object 12, 18, 24 will be light. As this pattern goes on, we can express the mode as a periodic function. Using the fact that sine and cosine functions are good examples of periodic functions, we can try to set the amplitudes of this mode to be sine or cosine functions.

See Fig.2. x-axis denotes n and y-axis denotes the amplitudes of Fig.1. Here, we see that the amplitudes are enveloped inside a sine function. Now, let's prove that such ones are good modes. First, we naturally have:

$$M\ddot{x}_n = -K(x_n - x_{n-1}) - K(x_n - x_{n+1}) \tag{1}$$

Now, let the amplitudes in our mode be the following:

$$x_n = A\sin(nkL + \phi_1)\sin(\omega t + \phi_2) \tag{2}$$

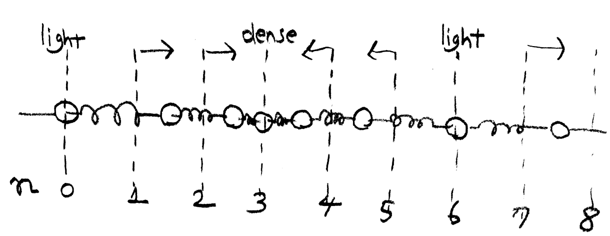

Figure 1: infinitely many objects and springs

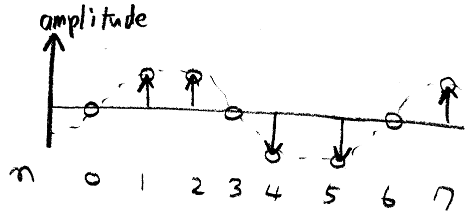

Figure 2: amplitudes of a given mode

where L is the distance between the neighboring object when the spring is neither stretched nor compressed. We include this factor for future convenience.

Then, letting $nkL + \phi_1 = \theta$ for simplicity, and plugging (2) to (1) yields:

$$-M\omega^2 A \sin\theta = -KA(2\sin\theta - \sin(\theta - kL) - \sin(\theta + kL)) \tag{3}$$

$$-M\omega^2 A \sin\theta = -KA\sin\theta(2 - 2\cos(kL)) \tag{4}$$

$$\omega^2 = \frac{2K}{M}(1 - \cos(kL)) = \frac{4K}{M}\sin^2\left(\frac{kL}{2}\right) \tag{5}$$

$$\omega = 2\sqrt{\frac{K}{M}}\sin\left(\frac{kL}{2}\right) \tag{6}$$

As this expression doesn't involve n, we see that it is a good mode oscillating altogether with the same frequency that doesn't depend on n. Also, in the example of our mode, we assumed certain x_ns were zero such as $n = 0, 3, 6$, but in general case there is no reason to assume so, as long as (2) is satisfied. We just assumed so to help you understand better by presenting the concepts of "dense" and "light." Furthermore, notice that the above equation is satisfied for any k. This is natural, as there are infinitely many modes as there are infinitely many objects.

Final comment. Even though we have considered springs in this article, our formulation here is general enough to be applied to most of the systems that involve travelling waves. For example, when the sound propagates, it has a pattern that dense part and light part are repeated. Actually, in the next article, we will derive the speed of sound using our formulation in this article.

Problem 1. Obtain how fast the spring waves propagate by using (6). You will see that our earlier inclusion of the factor L is now convenient. Also, approximate your answer in the limit when the distance between two adjacent objects (i.e. L) is much smaller than the wavelength of the spring wave. You will see that the answer won't depend on k any more in this limit.

You just obtained the speed of the spring wave in the limit L is much smaller than wavelength, but there is another, perhaps, much easier way to obtain the same expression. To this end, let's re-write (1) in a slightly different notation to avoid a confusion. Let's denote the displacement by ϕ instead of x.

$$M\ddot{\phi}_n = -K(\phi_n - \phi_{n-1}) - K(\phi_n - \phi_{n+1}) \tag{7}$$

$$= \quad -KL\frac{\phi_n - \phi_{n-1}}{L} + KL\frac{\phi_{n+1} - \phi_n}{L} \tag{8}$$

Recall now that n in ϕ_n means the displacement of nth object. Recalling that the position of the nth object when the displacement is zero is $x = nL$, from now on we will use the notation $\phi(x = nL) \equiv \phi_n$. Then, the above expression becomes

$$M\ddot{\phi}(nL) = -KL\frac{\phi(nL) - \phi(nL - L)}{L} + KL\frac{\phi(nL + L) - \phi(nL)}{L} \tag{9}$$

when L is infinitesimally small, we can write

$$\frac{\phi(nL) - \phi(nL - L)}{L} = \frac{\partial\phi}{\partial x}(nL - L/2), \qquad \frac{\phi(nL + L) - \phi(nL)}{L} = \frac{\partial\phi}{\partial x}(nL + L/2) \tag{10}$$

Thus, (9) becomes

$$M\frac{\partial^2\phi}{\partial t^2}(nL) \quad = \quad KL\left(\frac{\partial\phi}{\partial x}(nL + L/2) - \frac{\partial\phi}{\partial x}(nL - L/2)\right) \tag{11}$$

$$M\frac{\partial^2\phi}{\partial t^2}(nL) \quad = \quad KL^2\frac{\partial^2\phi}{\partial x^2}(nL) \tag{12}$$

which is exactly the partial differential equation for wave!

Problem 2. Find the speed of wave from (12) and check that it agrees with the answer in Problem 1 in the limit when L goes to zero.

Summary

- When objects of the same mass are connected consecutively by springs of the same spring constant, the amplitudes of each object given an eigenmode is given by a sine function.
- For such eigenmodes, one can easily calculate the group velocity. In the limit that the spacing between each object (i.e. the length of each spring) is zero, the group velocity is a constant that does not depend on the wave number (or equivalently, the wavelength).