Travelling wave

In our earlier article "What is Fourier series?," we have seen that the graph for sine and cosine functions looked wiggly. This wiggliness suggests that it could represent a wave well, since a wave is also wiggly and oscillating. To this end, consider the following function:

$$y = A\sin(kx - \omega t) \tag{1}$$

where x denotes distance and t denotes time. See Fig.1 for the graph of this function at a fixed t. y is called "displacement" and A is called "amplitude." Amplitude is the maximum displacement for a sine or cosine wave. We also see that the graph is repeating itself. We call the distance between each repetition "wavelength" and usually denote it by λ. See Fig.1 again. Now, Let's find the wavelength. We know that sine or cosine functions repeat themselves if argument is added by 2π. This consideration leads to following:

$$\begin{aligned} \sin(kx - \omega t) &= \sin(k(x + \lambda) - \omega t) \\ &= \sin(kx + k\lambda - \omega t) = \sin(kx - \omega t + 2\pi) \end{aligned} \tag{2}$$

So, we conclude $k\lambda = 2\pi$. In other words, $\lambda = 2\pi/k$ or $k = 2\pi/\lambda$.

Now, we have calculated the distance between each repetition but not the time between each repetition yet. The latter is called "period," and usually denoted by T. Let's calculate it. As before, we have:

$$\begin{aligned} \sin(kx - \omega t) &= \sin(kx - \omega(t + T)) \\ &= \sin(kx - \omega t - \omega T) = \sin(kx - \omega t - 2\pi) \end{aligned} \tag{3}$$

Therefore, we conclude $\omega T = 2\pi$. In other words, $T = 2\pi/\omega$ or $\omega = 2\pi/T$.

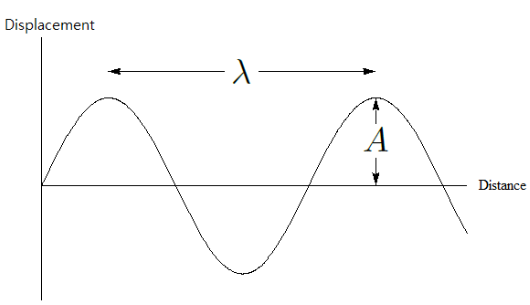

Figure 1: wave

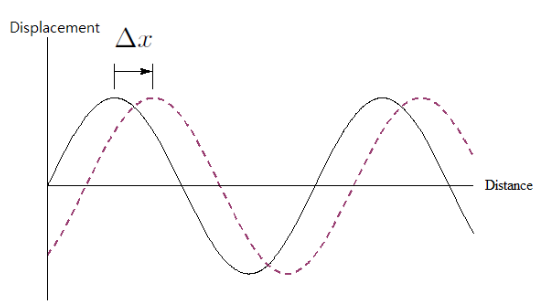

Figure 2: moving wave

Now, what would be "v" the speed of the wave? See Fig.2. The wave moves in x-direction. The solid line denotes the wave when $t = 0$ and the dotted line when $t = \Delta t$. During this interval Δt, the wave moves by Δx as you can easily see in the figure. Notice that the wave retains its form (i.e. displacement) as it moves. This consideration leads to the following formula:

$$\begin{aligned}
\sin(kx - \omega t) &= \sin(k(x + \Delta x) - \omega(t + \Delta t)) \\
kx - \omega t &= k(x + \Delta x) - \omega(t + \Delta t) \\
0 &= k\Delta x - \omega \Delta t \\
v = \frac{\Delta x}{\Delta t} &= \frac{\omega}{k} \qquad (4)
\end{aligned}$$

In other words, the velocity of the wave is given by ω/k. Notice that it is positive, since it is moving in positive x-direction. For example, if we considered the wave of $\sin(kx + \omega t)$, the same consideration would lead to following:

$$\frac{\Delta x}{\Delta t} = -\frac{\omega}{k} \qquad (5)$$

There is another way to derive the speed of wave. A wave takes T seconds (i.e. period) to move λ meters (i.e. wavelength). Therefore, we have:

$$v = \frac{\lambda}{T} = \frac{2\pi/k}{2\pi/\omega} = \frac{\omega}{k} \qquad (6)$$

Problem 1. Recall from our earlier article "Light as waves" that "frequency" f is defined by $f = 1/T$. Then, what is the relation between ω for the wave in (1) and its frequency f?

Summary

- $y = A\sin(kx - \omega t)$
  y is the displacement, A the amplitude, x distance, and t time.
- k satisfies $k\lambda = 2\pi$ where λ is the wavelength.
  ω satisfies $\omega T = 2\pi$ where T is the period.