## What are trigonometric functions?

Let's consider a right triangle as in Fig. 1. Given that the angle between $\overline{B C}$ and $\overline{A B}$ is $\theta$, what are the relationships between $a, b$ and $c$ ? See Fig. 2 and 3. Certainly, if $a$ is doubled or tripled, $b$ and $c$ will be also doubled or tripled, as long as $\theta$ is fixed. Therefore, we can easily see that the ratios between $a, b$, and $c$ remain fixed as long as $\theta$ doesn't change.

There are three ratios that can be made out of $a, b$, and $c$, namely $a / c$, $b / c, b / a$. These three ratios only depend on $\theta$. Therefore, we define them as follows:

$$
\begin{align*}
\cos \theta & =a / c  \tag{1}\\
\sin \theta & =b / c  \tag{2}\\
\tan \theta & =b / a \tag{3}
\end{align*}
$$

Therefore, if we know the value of these functions for a given angle $\theta$, we can know the ratios between the three sides of the right angle of which one of the angles is $\theta$. These functions are called "trigonometric functions," and we read "cos" as "cosine," "sin" as "sine" and "tan" as "tangent."

Before calculating these functions for specific values, let's find some identities that these trigonometric functions must satisfy.

First of all, we can easily see the following:

$$
\begin{equation*}
\tan \theta=\frac{b}{a}=\frac{b / c}{a / c}=\frac{\sin \theta}{\cos \theta} \tag{4}
\end{equation*}
$$

Therefore, we conclude:

$$
\begin{equation*}
\tan \theta=\frac{\sin \theta}{\cos \theta} \tag{5}
\end{equation*}
$$

Secondly, from Pythagoras' theorem, we have the following:

$$
\begin{gather*}
a^{2}+b^{2}=c^{2}  \tag{6}\\
a^{2} / c^{2}+b^{2} / c^{2}=c^{2} / c^{2}=1  \tag{7}\\
(a / c)^{2}+(b / c)^{2}=1 \tag{8}
\end{gather*}
$$

Therefore, we conclude:

$$
\begin{equation*}
\cos ^{2} \theta+\sin ^{2} \theta=1 \tag{9}
\end{equation*}
$$

which you must memorize. Here, I want to note that $(\cos \theta)^{2}$ is conventionally denoted as $\cos ^{2} \theta$, not as $\cos \theta^{2}$ which means $\cos \left(\theta^{2}\right)$. This also applied to sine and tangent.

Thirdly, see Fig. 4 and Fig. 5. Fig. 4 is essentially the same as Fig. 1. The only difference is we have noted that the angle between $\overline{A B}$ and $\overline{A C}$ is $90-\theta$. If we rotate this triangle as in Fig.5, it is easy to see that the following must hold:

$$
\begin{gather*}
\cos \left(90^{\circ}-\theta\right)=b / c=\sin \theta  \tag{10}\\
\sin \left(90^{\circ}-\theta\right)=a / c=\cos \theta  \tag{11}\\
\tan \left(90^{\circ}-\theta\right)=a / b=1 /(b / a)=1 / \tan \theta \tag{12}
\end{gather*}
$$

Therefore, we conclude:

$$
\begin{align*}
& \cos \left(90^{\circ}-\theta\right)=\sin \theta  \tag{13}\\
& \sin \left(90^{\circ}-\theta\right)=\cos \theta  \tag{14}\\
& \tan \left(90^{\circ}-\theta\right)=\frac{1}{\tan \theta} \tag{15}
\end{align*}
$$

Now, let's delve into calculating the trigonometric functions for $\theta=$ $30^{\circ}, 45^{\circ}$, and $60^{\circ}$. The trigonometric functions for generic angles are rather tricky, and you must rely on a calculator in most cases.

Firstly, let's consider the cases $\theta=30^{\circ}$ and $\theta=60^{\circ}$. If you have an equilateral triangle as in Fig. 6, you can easily see that you can divide it into two right triangles as in the picture. From this picture, we can immediately see the following:

$$
\begin{equation*}
\sin 30^{\circ}=\cos 60^{\circ}=(c / 2) / c=1 / 2 \tag{16}
\end{equation*}
$$

To obtain $\cos 30^{\circ}$ and $\sin 60^{\circ}$, we can use formula (9). The explicit calculation is left as an exercise for the readers. You must obtain

$$
\begin{equation*}
\cos 30^{\circ}=\sin 60^{\circ}=\frac{\sqrt{3}}{2} \tag{17}
\end{equation*}
$$

To obtain $\tan 30^{\circ}$ and $\tan 60^{\circ}$, we can use formula (5). We get:

$$
\begin{gather*}
\tan 30^{\circ}=\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}  \tag{18}\\
\tan 60^{\circ}=\sqrt{3} \tag{19}
\end{gather*}
$$

Finally, let's consider the case $\theta=45^{\circ}$. See Fig. 7. It's easy to see that the length of $\overline{A C}$ must be $a$, if the length of $\overline{B C}$ is $a$. Furthermore, by using Pythagoras' theorem, you can easily see that the length of $\overline{A B}$ must be $\sqrt{2} a$. Therefore we conclude:


Fig. 2


Fig. 3


Fig. 4
Fig. 5


Fig. 6


Fig. 7

$$
\begin{equation*}
\cos 45^{\circ}=\sin 45^{\circ}=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2} \tag{20}
\end{equation*}
$$

Also,

$$
\begin{equation*}
\tan 45^{\circ}=a / a=1 \tag{2}
\end{equation*}
$$

Problem 1. See Fig. 1. Express $c$ (the length of $\overline{A B}$ ) and $b$ (the length of $\overline{A C}$ ) in terms of $a$ (the length of $\overline{B C}$ ) and the trigonometric functions of $\theta$.

Problem 2. Let $x=\tan \theta$. If you correctly solved Problem 1, you will get $b=a \tan \theta=a x$. Using $c^{2}=a^{2}+b^{2}$, (1) and (2), express $\sin \theta$ and $\cos \theta$ in terms of $x$. This problem is useful to solve a problem in our later article "Integration by substitution" which in turn is useful to solve a problem in our later article "Another example of differential equation." It is interesting that the trigonometric functions appear in algebraic problems that have nothing to do with geometry.

Problem 3. Cosecant, secant, cotangent functions are defined by

$$
\begin{equation*}
\csc x=\frac{1}{\sin x}, \quad \sec x=\frac{1}{\cos x}, \quad \cot x=\frac{1}{\tan x} \tag{22}
\end{equation*}
$$

Given this, prove

$$
\begin{equation*}
1+\cot ^{2} x=\csc ^{2} x \tag{23}
\end{equation*}
$$

and

$$
\begin{equation*}
1+\tan ^{2} x=\sec ^{2} x \tag{24}
\end{equation*}
$$

You do not need to remember (23) and (24), but you must be able to derive them once provided with (22). (Hint ${ }^{1}$ )

Problem 4. See the figure. Consider an isosceles triangle with two sides $A$ and two angles $\theta$. What is the other side in terms of $A$ and $\theta$ ? (Hint ${ }^{2}$ )

[^0]

## Summary

- In a right triangle, the ratios between the three sides are given by $\sin \theta$, $\cos \theta, \tan \theta$. They are called trigonometric functions.
- See Fig. 1.

$$
\sin \theta=\frac{b}{c}, \quad \cos \theta=\frac{a}{c}, \quad \tan \theta=\frac{b}{a}
$$

- $\tan \theta=\frac{\sin \theta}{\cos \theta}$.
- From the Pythagorean theorem, we have $\sin ^{2} \theta+\cos ^{2} \theta=1$.
- $\cos \left(90^{\circ}-\theta\right)=\sin \theta, \quad \sin \left(90^{\circ}-\theta\right)=\cos \theta, \quad \tan \left(90^{\circ}-\theta\right)=\frac{1}{\tan \theta}$
- $\cos 60^{\circ}=\sin 30^{\circ}=\frac{1}{2}$
- $\sin 45^{\circ}=\cos 45^{\circ}=\frac{1}{\sqrt{2}}$
- $\tan 45^{\circ}=1$.


[^0]:    ${ }^{1}$ Use either (5) and (9). Or, alternatively, you can use the result of Problem 2.
    ${ }^{2}$ Divide the isosceles triangle into two right triangles.

