Vierbein formalism and Palatini action in general relativity

Youngsub Yoon

Abstract

We present the vierbein formalism and Palatini action in general relativity. The aim is to provide prerequisites for the Ashtekar variables formalism and the newer variables formalism for general relativity. This article should be accessible to students who are familiar with general relativity and differential forms.

1 Introduction of vierbein

The vierbein e^a_μ is defined by the following relation:

$$g_{\mu\nu} = \eta_{ab} e^a_\mu e^b_\nu \tag{1}$$

where $g_{\mu\nu}$ is the metric and η_{ab} is defined as follows

$$\eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{2}$$

In other words, η_{ab} is the metric for the flat Cartesian coordinate. *Vier* means four in German and is pronounced as "fear." *Bein* means leg in German, and is pronounced as "bine." We can see here that vierbein is like the "square root" of the metric, as "square" of e is the metric. We also see here that (1) doesn't uniquely determine the vierbein. Considering the symmetricity of the metric (i.e. $g_{\mu\nu} = g_{\nu\mu}$) the metric has ten independent components. In other words, (1) are ten equations. On the other hand, we have sixteen unknowns, as each of a and μ in e^a_μ can have four values. (i.e. $a = 0, 1, 2, 3$, $\mu = 0, 1, 2, 3$.) Therefore, there are six degrees of freedom in choosing the vierbein. (We will talk more about this freedom in the next article.) Nevertheless, different choices lead to the same physics.

(1) also implies the following:

$$\eta^{ab} = g^{\mu\nu} e^a_\mu e^b_\nu \tag{3}$$

where $g^{\mu\nu}$ is the inverse of the metric $g_{\mu\nu}$, and η^{ab} is the inverse of η_{ab}

We can prove this by multiplying $\eta_{bc}e^{c}{}_{\lambda}$ on its both-hand sides as follows.

$$\begin{aligned}\eta^{ab}\eta_{bc}e^{c}_{\lambda} &= g^{\mu\nu}e^{a}_{\mu}e^{b}_{\nu}\eta_{bc}e^{c}_{\lambda}\\ \delta^{a}_{c}e^{c}_{\lambda} &= g^{\mu\nu}e^{a}_{\mu}g_{\nu\lambda}\\ e^{a}_{\lambda} &= e^{a}_{\lambda}\end{aligned} \tag{4}$$

So, we conclude that the left-hand side of (3) is the same as its right-hand side.

Notice that vierbein e has two kinds of indices. One is the Latin index (i.e. $a, b\cdots$) and the other is the Greek index (i.e. $\mu, \nu\cdots$). We raise and lower the Latin indices by η^{ab} or η_{ab}, as they have two Latin indices and we raise and lower the Greek indices by $g^{\mu\nu}$ or $g_{\mu\nu}$ as they have two Greek indices. In this case, the Latin indices are called "Lorentz indices," and the Greek indices "spacetime indices." Now we can freely raise and lower the indices of the vierbein as follows:

$$e^{a\nu} = g^{\mu\nu}e^{a}_{\mu} \tag{5}$$

$$e_{b\mu} = \eta_{ab}e^{a}_{\mu} \tag{6}$$

$$e^{\nu}_{b} = g^{\mu\nu}\eta_{ab}e^{a}_{\mu} \tag{7}$$

It turns out that e^{ν}_{b} defined above is the inverse of e^{a}_{μ}. One can check this as follows:

$$\begin{aligned}e^{\nu}_{b}e^{b}_{\lambda} &= g^{\mu\nu}\eta_{ab}e^{a}_{\mu}e^{b}_{\lambda} = \delta^{\nu}_{\lambda}\\ e^{\nu}_{b}e^{c}_{\nu} &= g^{\mu\lambda}\eta_{ab}e^{a}_{\mu}e^{c}_{\lambda} = \delta^{c}_{b}\end{aligned}$$

where we have used (1) and (3).

So, why is vierbein called vierbein? First, notice

$$\eta_{ab} = g_{\mu\nu}e^{\mu}_{a}e^{\nu}_{b} \tag{8}$$

If we regard e^{μ}_{a} as a four-vector labeled by a, then we can write:

$$\eta_{ab} = \vec{e}_a \cdot \vec{e}_b \tag{9}$$

Thus, vierbein is a set of four orthonormal vectors (i.e., $\vec{e}_0$, $\vec{e}_1$, $\vec{e}_2$, $\vec{e}_3$) defined at each point. In other words, vierbein can be viewed as four legs attached at each point in spacetime. Notice also that (9) is exactly Minkwoski version of (5) in "Dimension of orthogonal group." Thus, we see that the degree of freedom at each point must be the dimension of the Lie group $SO(3,1)$, which is 6 as we found earlier.

2 Spin connection

Given the definition of vierbein, let's turn to the covariant partial derivative D_μ that acts on both spacetime indices and Lorentz indices. As much as one requires $\nabla_\alpha g_{\mu\nu} = 0$

as a sacred characteristic of the metric, we require the following as a sacred characteristic of the vierbein:

$$D_\mu e_{b\nu} = 0 \tag{10}$$

Furthermore, notice that there are two different types of indices, so to define expression above we need to use two different types of connection, namely the Christoffel symbols Γ and the “spin connection”ω as follows:

$$D_\mu e_{b\nu} = \partial_\mu e_{b\nu} - \Gamma^\alpha_{\mu\nu} e_{b\alpha} - \omega^c_{\mu b} e_{\nu c} = 0 \tag{11}$$

Noticing that the first two terms together make up the ordinary covariant derivative for spacetime indices, we can write:

$$\begin{aligned} \omega^c_{\mu b} e_{\nu c} &= \nabla_\mu e_{b\nu} \\ e^\nu_a \omega^c_{\mu b} e_{\nu c} &= e^\nu_a \nabla_\mu e_{b\nu} \end{aligned}$$

which implies

$$\omega_{\mu ab} = e^\nu_a \nabla_\mu e_{b\nu} \tag{12}$$

Now, observe the following:

$$\begin{aligned} \nabla_\mu \eta_{ab} &= 0 \\ \nabla_\mu (e^\nu_a e_{b\nu}) &= 0 \\ e^\nu_a \nabla_\mu e_{b\nu} + e_{b\nu} \nabla_\mu e^\nu_a &= 0 \\ e^\nu_a \nabla_\mu e_{b\nu} + e^\nu_b \nabla_\mu e_{a\nu} &= 0 \end{aligned}$$

where we have used the fact $\nabla_\mu g_{\alpha\beta} = 0$ in the last line.

Thus we can conclude:

$$\begin{aligned} \omega_{\mu ab} &= e^\nu_a \nabla_\mu e_{b\nu} \\ &= -e^\nu_b \nabla_\mu e_{a\nu} \\ &= -\omega_{\mu ba} \end{aligned} \tag{13}$$

In other words, the spin connection is antisymmetric with respect to the two Lorentz-indices.

3 Torsion two-form

Observe the following:

$$D_\mu e^b_\nu = \partial_\mu e^b_\nu - \Gamma^\alpha_{\mu\nu} e^b_\alpha + \omega^b_{\mu c} e^c_\nu = 0 \tag{14}$$

$$D_\nu e^b_\mu = \partial_\nu e^b_\mu - \Gamma^\alpha_{\nu\mu} e^b_\alpha + \omega^b_{\nu c} e^c_\mu = 0 \tag{15}$$

By subtraction, we immediately see the following:

$$\partial_\mu e^b_\nu - \partial_\nu e^b_\mu + \omega^b_{\mu c} e^c_\nu - \omega^b_{\nu c} e^c_\mu = (\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}) e^b_\alpha \tag{16}$$

If we use the differential form notation as follows:

$$e^b \equiv e^b_\nu dx^\nu \tag{17}$$

$$\omega^b{}_c \equiv \omega^b_{\nu c} dx^\nu \tag{18}$$

and define the torsion tensor as follows:

$$T^b \equiv \frac{1}{2} T^b_{\mu\nu} dx^\mu \wedge dx^\nu \equiv \frac{1}{2}[(\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}) e^b_\alpha] dx^\mu \wedge dx^\nu \tag{19}$$

we obtain:

$$de^b + \omega^b{}_c \wedge e^c = T^b \tag{20}$$

In the index-free notation, we can write the above equation as follows:

$$de + \omega \wedge e = T \tag{21}$$

4 Curvature two-form

In this section, we will obtain an expression for the curvature two-form, the vierbein analogue of the Riemann tensor. First, as we have:

$$(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) e_{d\rho} = R_{\rho\sigma\mu\nu} e^\sigma_d + (\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}) \nabla_\alpha e_{d\rho} \tag{22}$$

we can write as follows:

$$\begin{aligned} R_{cd\mu\nu} &\equiv R_{\rho\sigma\mu\nu} e^\rho_c e^\sigma_d \\ &= e^\rho_c [(\nabla_\mu \nabla_\nu - \nabla_\nu \nabla_\mu) e_{d\rho} - (\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}) \nabla_\alpha e_{d\rho}] \end{aligned} \tag{23}$$

However, we have:

$$\begin{aligned} & e^\rho_c \nabla_\mu \nabla_\nu e_{d\rho} \\ &= \nabla_\mu (e^\rho_c \nabla_\nu e_{d\rho}) - (\nabla_\mu e^\rho_c)(\nabla_\nu e_{d\rho}) \end{aligned} \tag{24}$$

Furthermore, we have:

$$\begin{aligned} (\nabla_\mu e^\rho_c)(\nabla_\nu e_{d\rho}) &= (\nabla_\mu e^\alpha_c) \delta^\rho_\alpha (\nabla_\nu e_{d\rho}) \\ &= (\nabla_\mu e^\alpha_c) \eta^{ab} e^\rho_a e_{b\alpha} (\nabla_\nu e_{d\rho}) \end{aligned} \tag{25}$$

Therefore, we can write:

$$\begin{aligned} R_{cd\mu\nu} &= \nabla_\mu \omega_{\nu cd} - \nabla_\nu \omega_{\mu cd} - \eta^{ab}(\omega_{\mu bc}\omega_{\nu ad} - \omega_{\nu bc}\omega_{\mu ad}) - (\Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu})\omega_{\alpha cd} \\ &= \partial_\mu \omega_{\nu cd} - \partial_\nu \omega_{\mu cd} + \eta^{ab}(\omega_{\mu cb}\omega_{\nu ad} - \omega_{\nu cb}\omega_{\mu ad}) \end{aligned} \tag{26}$$

Raising one of the indices, we can write:

$$\begin{aligned} R^c{}_{d\mu\nu} &= \partial_\mu \omega^c_{\nu d} - \partial_\nu \omega^c_{\mu d} + \omega^c_{\mu b}(\eta^{ab}\omega_{\nu ad}) - \omega^c_{\nu b}(\eta^{ab}\omega_{\mu ad}) \\ &= \partial_\mu \omega^c_{\nu d} - \partial_\nu \omega^c_{\mu d} + \omega^c_{\mu b}\omega^b_{\nu d} - \omega^c_{\nu b}\omega^b_{\mu d} \end{aligned} \tag{27}$$

If we use the differential form notation, we have:

$$R^c{}_d \equiv \frac{1}{2} R^c{}_{d\mu\nu} dx^\mu \wedge dx^\nu = d\omega^c{}_d + \omega^c{}_b \wedge \omega^b{}_d \tag{28}$$

$$\equiv \frac{1}{2} R^c{}_{dab} e^a \wedge e^b \tag{29}$$

which is called the "curvature two-form." If you are familiar with Yang-Mills theory, you will notice that the above equation has exactly the same structure.

In index-free notation, we can write the above equation as follows:

$$R = d\omega + \omega \wedge \omega \tag{30}$$

Therefore, if the torsion vanishes, the right-hand side of (21) is zero, so we can easily obtain ω if e is given. From this ω, we can obtain the curvature by using (30)

5 Bianchi identities

Let's take the exterior derivative of (21). We get:

$$dT = d\omega \wedge e - \omega \wedge de \tag{31}$$

We also have:

$$\omega \wedge T = \omega \wedge de + \omega \wedge \omega \wedge e \tag{32}$$

Therefore, we conclude:

$$dT + \omega \wedge T = R \wedge e \tag{33}$$

If torsion vanishes, we simply have:

$$R \wedge e = 0 \tag{34}$$

It turns out that the above equation is the vierbein version of the following Bianchi identity:

$$R_{abcd} + R_{bcad} + R_{cabd} = 0 \tag{35}$$

(**Problem 1.** Show this.) Now, we have to derive the second Bianchi identity. From (30), we have:

$$\begin{aligned} dR &= d\omega \wedge \omega - \omega \wedge d\omega \\ \omega \wedge R &= \omega \wedge d\omega + \omega \wedge \omega \wedge \omega \\ R \wedge \omega &= d\omega \wedge \omega + \omega \wedge \omega \wedge \omega \end{aligned} \tag{36}$$

Therefore, we conclude:

$$dR + \omega \wedge R - R \wedge \omega = 0 \tag{37}$$

It also turns out that this is the vierbein version of the following Bianchi identity:

$$\nabla_a R_{bcd}{}^e + \nabla_b R_{cad}{}^e + \nabla_c R_{abd}{}^e = 0 \tag{38}$$

6 Gauge-covariant exterior derivative

It is natural to define the gauge-covariant exterior derivative as follows:

$$Du^b = du^b + \omega^b{}_c \wedge u^c \tag{39}$$

since we will be able to write (20) as:

$$T^b = De^b \tag{40}$$

and (33) as

$$DT^b = R^b{}_c \wedge e^b \tag{41}$$

Also, given (39), taking a similar step to the one that led to (33), we obtain:

$$D^2 u^b = R^b{}_c \wedge u^c \tag{42}$$

To go further, it may be useful to write the gauge-covariant exterior derivative in component form as follows:

$$D_\mu v^b = \partial_\mu v^b + \omega^b_{\mu d} v^d \tag{43}$$

which implies

$$\begin{aligned} D_\mu v_c &= \partial_\mu v_c - \omega^d_{\mu c} v_d \\ Dv_c &= dv_c - \omega^d{}_c \wedge v_d \end{aligned} \tag{44}$$

Then, as $R^b{}_c$ has one upper index (i.e. b) and one lower index (i.e. c), using (39) and (44), we have:

$$\begin{aligned} DR^b{}_c &= dR^b{}_c + \omega^b{}_d \wedge R^d{}_c - \omega^d{}_c \wedge R^b{}_d \\ &= dR^b{}_c + \omega^b{}_d \wedge R^d{}_c - R^b{}_d \wedge \omega^d{}_c \end{aligned} \tag{45}$$

where in the last step, we used the fact that $R^b{}_d$ is a two-form. Comparing with (37), we can conclude that the second Bianchi identity can be written as:

$$DR^b{}_c = 0 \tag{46}$$

7 Hodge dual

In four-dimensions, the Hodge-duality map, * (called the "star" operator), is defined as follows:

$$*(e^{a_1} \wedge \cdots e^{a_n}) \equiv \frac{1}{(4-n)!} \epsilon^{a_1 \cdots a_n}{}_{a_{n+1} \cdots a_4} e^{a_{n+1}} \wedge \cdots \wedge e^{a_4} \tag{47}$$

In other words, it is a natural and covariant map that provides a linear isomorphism between n-forms and $(4-n)$-forms. Here ϵ is Levi-Civita symbol defined via $\epsilon_{0123} = 1$, and indices are raised by η^{ab}. One can similarly define the star operator in any other dimensions.

To get a sense of what the star operator does, let me give you some examples:

$$*(e^0 \wedge e^1) = \epsilon^{01}{}_{23} e^2 \wedge e^3 = -e^2 \wedge e^3 \tag{48}$$

$$*(e^2 \wedge e^3) = \epsilon^{23}{}_{01} e^0 \wedge e^1 = e^0 \wedge e^1 \tag{49}$$

Take two n-forms as follows:

$$\lambda = \frac{1}{n!} \lambda_{a_1 \cdots a_n} e^{a_1} \wedge \cdots \wedge e^{a_n} \tag{50}$$

$$\sigma = \frac{1}{n!} \sigma_{b_1 \cdots b_n} e^{b_1} \wedge \cdots \wedge e^{b_n} \tag{51}$$

Then, given the definition of the Hodge star operator, we have:

$$\begin{aligned} \lambda \wedge *\sigma &= \frac{1}{(n!)^2} \lambda_{a_1 \cdots a_n} \sigma_{b_1 \cdots b_n} \frac{1}{(4-n)!} \epsilon^{b_1 \cdots b_n}{}_{a_{n+1} \cdots a_4} e^{a_1} \wedge \cdots e^{a_n} \wedge e^{a_{n+1}} \wedge \cdots \wedge e^{a_4} \\ &= \frac{1}{n!} \lambda_{a_1 \cdots a_n} \sigma_{b_1 \cdots b_n} \frac{\epsilon^{b_1 \cdots b_n}{}_{a_{n+1} \cdots a_4} \epsilon^{a_1 \cdots a_n a_{n+1} \cdots a_4}}{n!(4-n)!} e^0 \wedge e^1 \wedge e^2 \wedge e^3 \\ &= \frac{1}{n!} \lambda_{a_1 \cdots a_n} \sigma_{b_1 \cdots b_n} \eta^{a_1 b_1} \cdots \eta^{a_n b_n} e^0 \wedge e^1 \wedge e^2 \wedge e^3 \\ &= \frac{1}{n!} \lambda_{a_1 \cdots a_n} \sigma^{a_1 \cdots a_n} e^0 \wedge e^1 \wedge e^2 \wedge e^3 \end{aligned} \tag{52}$$

If we define the natural inner product as

$$< \lambda, \sigma >= \frac{1}{n!} \lambda_{a_1 \cdots a_n} \sigma^{a_1 \cdots a_n} \tag{53}$$

and ϵ, the volume form, as follows:

$$\epsilon = e^0 \wedge e^1 \wedge e^2 \wedge e^3 \tag{54}$$

then, we can write (52) as follows:

$$\lambda \wedge *\sigma =< \lambda, \sigma > \epsilon \tag{55}$$

The reason why ϵ is called the volume form is because it exactly gives the Jacobian factor or equivalently $\sqrt{-g}$. Let's prove this.

$$\begin{aligned}\epsilon = e^0 \wedge e^1 \wedge e^2 \wedge e^3 &= e^0_\mu e^1_\nu e^2_\rho e^3_\sigma dx^\mu \wedge dx^\nu \wedge dx^\rho \wedge dx^\sigma \\ &= e^0_\mu e^1_\nu e^2_\rho e^3_\sigma \epsilon^{\mu\nu\rho\sigma} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \\ &= (\det e) dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \end{aligned} \tag{56}$$

Now, we have to prove that $\det e$ is the Jacobian factor.

Recall:

$$g_{\mu\nu} = e^I_\mu e^J_\nu \eta_{IJ} \tag{57}$$

In the matrix notation this would be:

$$g = e^T \eta e \tag{58}$$

Taking the determinant, we have:

$$\det g = \det e \det \eta \det e = -(\det e)^2 \tag{59}$$

So, we conclude:

$$\det e = \sqrt{-g} \tag{60}$$

This completes the proof that ϵ is the volume form.

Actually, the Hodge dual can be defined without resorting to the vierbein. For example, in the case that the spacetime concerned is 4-dimensional, the hodge dual can be defined as follows:

$$*(dx^{a_1} \wedge \cdots dx^{a_n}) \equiv \frac{1}{(4-n)!} \epsilon^{a_1 \cdots a_n}{}_{a_{n+1} \cdots a_4} dx^{a_{n+1}} \wedge \cdots \wedge dx^{a_4} \tag{61}$$

where ϵ here is the Levi-Civita tensor. (By abuse of notation, we used the same letter for both Levi-Civita symbol and Levi-Civita tensor.) Compare this with our earlier formula (47). The Levi-Civita symbol is replaced by the Levi-Civita tensor. The hodge dual defined above is equivalent to our earlier definition. Let's check this by considering the following. According to our earlier definition, we have

$$*1 = e^0 \wedge e^1 \wedge e^2 \wedge e^3 \tag{62}$$

whereas according to our new definition, we have

$$*1 = \sqrt{-g} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 \tag{63}$$

So (62) and (63) are same. Here, in this article, we will not prove the equivalence of (47) and (61) in the general case.

As an aside, we want to note that our paper with Brian Kong "Black hole entropy and Hawking radiation spectrum predictions without Immirzi parameter" is based on the observation that traditional loop quantum gravity used the Levi-Civita symbol where the Levi-Civita tensor should be used; for spacetime indices the Levi-Civita tensor, not the Levi-Civita symbol, should be used.

8 The Palatini action

Now, consider the following integral:

$$\int \epsilon_{IJKL}\frac{1}{2}e^I \wedge e^J \wedge R^{KL} \tag{64}$$

Notice that R^{KL} here is not the Ricci tensor, but a curvature two-form. Notice also that $R^{KL} = -R^{LK}$. We have:

$$\begin{aligned}
&= \int R^{KL} \wedge (\frac{1}{2}\epsilon_{IJKL}e^I \wedge e^J) \\
&= \int R^{KL} \wedge *(e_K \wedge e_L) \\
&= \int \frac{1}{2}R^{KL}{}_{MN}e^M \wedge e^N \wedge *(e_K \wedge e_L) \\
&= \int \frac{1}{2}R^{KL}{}_{MN} < e^M \wedge e^N, e_K \wedge e_L > \\
&= \int \frac{1}{2}R^{KL}{}_{MN}(\delta^M_K\delta^N_L - \delta^M_L\delta^N_K)\epsilon \\
&= \int R\epsilon
\end{aligned} \tag{65}$$

So, this is precisely the Einstein-Hilbert action! Therefore, we conclude:

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R = \frac{1}{16\pi G}\int \epsilon_{IJKL}\frac{1}{2}e^I \wedge e^J \wedge R^{KL} \tag{66}$$

Summary

- $g_{\mu\nu} = \eta_{ab}e^a_\mu e^b_\nu$ where $g_{\mu\nu}$ is the metric and η_{ab} is the metric for the flat Cartesian coordinate. e^a_μ is called "vierbein."
- In other words, vierbein is like the "square root" of the metric.
- We raise and lower the Lorentz indices by η and we raise and lower the spacetime indices by g.
- $D_\mu e_{b\nu} = 0$. Unlike ∇_μ, the new partial derivatives D_μ act on the Lorentz indices as well.

$$D_\mu e_\nu \approx \partial_\mu e_\nu - \Gamma_{\mu\nu}e - \omega_\mu e_\nu = 0$$

- The spin connection is antisymmetric with respect to the two Lorentz-indices $\omega_{\mu ab} = -\omega_{\mu ba}$.
- $de + \omega \wedge e = T$.
- $d\omega + \omega \wedge \omega = R$.
- $R \wedge e = 0$.
- Gauge-covariant exterior derivative is given by

$$Du^b = du^b + {\omega^b}_c \wedge u^c$$

- In four-dimensions, the Hodge-duality map, $*$ gives a natural and covariant map between n-forms and $(4-n)$-forms as follow.

$$*(e^{a_1} \wedge \cdots e^{a_n}) \equiv \frac{1}{(4-n)!} {\epsilon^{a_1 \cdots a_n}}_{a_{n+1} \cdots a_4} e^{a_{n+1}} \wedge \cdots \wedge e^{a_4}$$

- The volume form is given by $\epsilon = e^0 \wedge e^1 \wedge e^2 \wedge e^3$.
- The Einstein-Hilbert action is given by

$$\int \epsilon_{IJKL} \frac{1}{2} e^I \wedge e^J \wedge R^{KL}$$

Further Reading

The following books and lecture notes which I recommend were most helpful when preparing this review paper: *General Relativity* by Robert Wald, *Geometry, Topology and Physics* by M. Nakahara.*Gravitation, Gauge Theories and Differential Geometry* by Tohru Eguchi, Peter B. Gilkey and Andrew J. Hanson, *Introductory lectures to loop quantum gravity* by Pietro Doná and Simone Speziale (arXiv:1007.0402), *Ashtekar Variables in Classical General Relativity* by Domenico Giulini (arXiv: gr-qc/9312032), *Quantum Gravity* by Carlo Rovelli.