What is representation?

Recall what a group is. To qualify as a group, the set and operation, $(G, \bullet)$, must satisfy following four requirements known as the group axiom.

(Closure) If f and g are in G then $h = f \bullet g$ is always in G.
(Associativity) If f, g and h are in G then $f \bullet (g \bullet h) = (f \bullet g) \bullet h$ is always satisfied.
(Identity element) There exists an element e in G such that for every element f in G, $e \bullet f = f \bullet e = f$ is satisfied.
(Inverse element) For every element f in G, there exists an inverse f^{-1} such that $f \bullet f^{-1} = f^{-1} \bullet f = e$

Now we want to represent a group using matrices. A representation of G is a mapping, D of the elements of G onto a set of matrices with the following properties.

1. $D(e) = I$ where I is the identity matrix.
2. $D(g_1)D(g_2) = D(g_1 \bullet g_2)$

As an aside, we want to remark that there is a field in mathematics called "representation theory." Apparently, mathematicians study how to represent groups by matrices.

Problem 1. Show that the group we considered as an example in "What is a group?" (i.e. integers with addition) can be represented as follows:

$$D(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tag{1}$$

Problem 2. Show that $D(g) = I$ for any g in group G is a representation. Of course, it goes without saying that such a representation is useless. It is called "trivial representation," and exists for any group.

Problem 3. Let D be a representation of a group G, and g, g_1, g_2, g_3 some of their elements. Show $D(g_1)D(g_2)D(g_3) = D(g_1 \bullet g_2 \bullet g_3)$ and $D(g^{-1}) = (D(g))^{-1}$.

Problem 4. Show that the set of invertible $n \times n$ matrices forms a group if the group action $\bullet$ is defined to be ordinary matrix multiplication. Such a group is called "the general linear group of n" and denoted by $GL(n, \mathbf{R})$ if the entries in the matrix are real numbers and $GL(n, \mathbf{C})$ if the entries in the matrix are complex numbers.

Problem 5. Show that the set of $n \times n$ matrices with determinant 1 forms a group if the group action $\bullet$ is defined to be ordinary matrix multiplication. Such a group is called "the special linear group of n" and denoted by $SL(n, \mathbf{R})$ if the entries in the matrix are real numbers and $SL(n, \mathbf{C})$ if the entries in the matrix are complex numbers. As an aside $SL(2, \mathbf{C})$ describes what is called "Möbius transformation" (Yes, the same Möbius as in Möbius strip) and plays a very important role in string theory.

Problem 6. Show that $SU(N)$, which is the $N \times N$ unitary matrix with determinant 1, $SO(N)$,which is the $N \times N$ orthogonal matrix with determinant 1, and $SO(1, 3)$ form groups respectively. The latter is known as the Lorentz group for an obvious reason. We will talk more about it in our later article "The Lorentz group and its representations."

Summary

- Matrices can be used to represent a group.