Why is the probability proportional to the wave function squared?

In our earlier articles, we mentioned that a probability is proportional to the wave function squared (precisely speaking, its absolute value squared, or the absolute value of the coefficient squared). In this article, we will explain why. We will consider a simple scattering problem and show that the probability of reflection and the probability of transmission add up to 1 in such a case.

To this end, we first need to explain that the first space derivative of the solution to the time-independent Schrödinger equation is always continuous except at the points where the potential is infinity or negative infinity. Let's show this. The Schrödinger equation is given by

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x) \tag{1}$$

What we want to show is

$$\lim_{\epsilon\to 0}\left(\frac{\partial\psi}{\partial x}\bigg|_{x=a+\epsilon} - \frac{\partial\psi}{\partial x}\bigg|_{x=a-\epsilon}\right) = 0 \tag{2}$$

as long as $-\infty < V(a) < \infty$.

From (1), we have

$$\begin{aligned}
\frac{\partial^2\psi(x)}{\partial x^2} &= -\frac{2m}{\hbar^2}\left(E - V(x)\right)\psi(x) &(3)\\
\int_{a-\epsilon}^{a+\epsilon}\frac{\partial^2\psi(x)}{\partial x^2}dx &= -\frac{2m}{\hbar^2}\int_{a-\epsilon}^{a+\epsilon}\left(E - V(x)\right)\psi(x)dx &(4)\\
\lim_{\epsilon\to 0}\left(\frac{\partial\psi}{\partial x}\bigg|_{x=a+\epsilon} - \frac{\partial\psi}{\partial x}\bigg|_{x=a-\epsilon}\right) &= -\frac{2m}{\hbar^2}\lim_{\epsilon\to 0}(E - V(a))\psi(a)2\epsilon &(5)
\end{aligned}$$

Thus, we obtain (2). (2) also implies that $\psi(x)$ is continuous at a, as (2) cannot be satisfied, otherwise.

Now, we are ready. We will solve the Schrödinger equation for the following potential for a particle with mass m:

$$V(x) = \begin{cases} 0, & x < 0 \\ V_0, & x \geq 0 \end{cases} \tag{6}$$

For $E > V_0 > 0$, the solution is given by

$$\psi(x) = \begin{cases} Ae^{ik_Lx} + Be^{-ik_Lx}, & x < 0 \\ Fe^{ik_Rx} + Ge^{-ik_Rx}, & x \geq 0 \end{cases} \tag{7}$$

where

$$\frac{\hbar^2 k_L^2}{2m} = E, \qquad \frac{\hbar^2 k_R^2}{2m} = E - V_0 \tag{8}$$

As $Ae^{ik_L x}$ is the eigenvector of the momentum operator with the eigenvalue $\hbar k_L$, A is related to the amplitude of the wave moving right with the momentum $\hbar k_L$ (i.e., the speed $\hbar k_L/m$) in the region $x < 0$. Similarly, B is related to the amplitude of the wave moving left with the speed $\hbar k_L/m$ in the region $x < 0$. Likewise, $F(G)$ is related to the amplitude of the wave moving right(left) with the speed $\hbar k_R/m$ in the region $x > 0$.

Given this, let's impose a physical condition. Let's say that we are shooting a particle with energy E from the left (i.e., $x < 0$). This particle moving right with the speed $\hbar k_L/m$ has the amplitude A. Then, it will either reflect at $x = 0$ and move left with the speed $\hbar k_L/m$ in the region $x < 0$ or transmit at $x = 0$ or move right with the speed $\hbar k_R/m$ in the region $x > 0$. The related amplitude is respectively, B and F. As there is no particle coming from the right for the region $x > 0$ (we didn't shoot any particle toward the left on the right region), we should have $G = 0$. Thus, (7) becomes

$$\psi(x) = \begin{cases} Ae^{ik_L x} + Be^{-ik_L x}, & x < 0 \\ Fe^{ik_R x}, & x \geq 0 \end{cases} \tag{9}$$

Now, we can impose the condition that ψ and $\dfrac{\partial \psi}{\partial x x}$ are continuous at $x = 0$, and obtain a relation between A, B and F.

At this point, careful readers may notice that our problem is exactly the same as the one in our earlier article "Reflection and transmission of travelling wave." The calculation is exactly the same as before and we obtain (Problem 1.)

$$\frac{B}{A} = \frac{k_L - k_R}{k_L + k_R}, \qquad \frac{F}{A} = \frac{2k_L}{k_L + k_R} \tag{10}$$

Given this, how can we calculate the probability that the particle will reflect at $x = 0$ and transmit at $x = 0$. The flux of particle moving right in the left region is $|A|^2(\hbar k_L/m)$. The flux of reflected particle is given by $|B|^2(\hbar k_L/m)$. The flux of transmitted particle is given by $|F|^2(\hbar k_R/m)$. Thus, the probability of reflection, R and the probability of transmission, T are given by

$$R = \frac{|B|^2(\hbar k_L/m)}{|A|^2(\hbar k_L/m)}, \qquad T = \frac{|F|^2(\hbar k_R/m)}{|A|^2(\hbar k_L/m)} \tag{11}$$

Problem 2. Simplify the above expressions for R and T by plugging in (10). Check that $R + T = 1$.

This completes our demonstration. Notice that we would not obtain this result, if the probability is proportional to the wave function cubed or to the fourth power. (Of course, the cubed case is already ruled out because it would give a negative probability.)

Max Born was the one who first showed that the probability must be proportional the wave function (precisely speaking absolute value) squared. He considered a similar case to the one we considered in this article. Instead of (6), he considered a potential non-zero only near $x = 0$. In this case, as we have $k_L = k_R$, (11) becomes

$$R = \frac{|B|^2}{|A|^2}, \qquad T = \frac{|F|^2}{|A|^2} \tag{12}$$

Then, he derived a relation similar to (2) by manipulating Schrödinger equation and showed $|A|^2 = |B|^2 + |F|^2$, which results in $R + T = 1$. Subsequently, he went on to consider 3 dimensional case to show that the probability is indeed proportional to the wave function squared.

Problem 3. Consider the case (6) again, but this time $0 < E < V_0$. Show that $R = 1$, i.e., total reflection. In other words, the particle is too weak to overcome the potential barrier (i.e., $E < V_0$) to reach $x = +\infty$. You can think of it as potential barrier but infinitely thick. The tunneling probability is therefore zero.

Summary

- Except at the points where the potential is $+\infty$ or $-\infty$, the solution to the time-independent Schrödinger equation and its first space derivative are always continuous.

- By considering a scattering problem, we can confirm that the probability is proportional to the absolute value of the wave function squared. Otherwise, the probability for reflection and the probability for transmission do not add up to 1.