Wilson line and Wilson loop

1 Revisiting gauge invariance

In this section, we re-visit gauge invariance to introduce the Wilson line and Wilson loop. In particular, we will focus on the non-Abelian gauge theory case, since it is more general than the Abelian case and the restriction to the Abelian case is quite trivial.

It has been already mentioned in "Non-Abelian gauge theory" that gauge invariance means that the theory is invariant under:

$$\psi(x) \to V(x)\psi(x) \tag{1}$$

where $V(x)$ is a unitary matrix. Explicitly, it can be written as $V(x) = e^{ig\theta^a(x)T^a}$. It was also mentioned that $\partial_\mu\psi$ does not transform covariantly under the above gauge transformation. In other words,

$$n^\mu \partial_\mu \psi = \lim_{\epsilon\to 0} \frac{1}{\epsilon}[\psi(x+\epsilon n) - \psi(x)] \tag{2}$$

doesn't transform covariantly. How can we fix this problem?

To this end, let's define a scalar quantity $U(y,x)$ that transforms as follows under the gauge transformation:

$$U(y,x) \to V(y)U(y,x)V^{-1}(x) \tag{3}$$

Also, we demand that $U(x,x) = 1$, which is consistent with the above transformation, since:

$$U(x,x) = 1 \to V(x)U(x,x)V^{-1}(x) = 1 \tag{4}$$

Now, it is easy to see that $\psi(y)$ and $U(y,x)\psi(x)$ have the same transformation law. Therefore, instead of the naive partial differentiation (i.e. $\partial_\mu\psi$) that doesn't respect gauge covariance, we can define the following "covariant derivative" which is meaningful:

$$n^\mu D_\mu \psi = \lim_{\epsilon\to 0} \frac{1}{\epsilon}[\psi(x+\epsilon n) - U(x+\epsilon n, x)\psi(x)] \tag{5}$$

Let's Taylor expand U. Remembering that $U(x,x)$ is 1, we can write:

$$U(x+\epsilon n, x) = 1 + ig\epsilon n^\mu A_\mu + O(\epsilon^2) \tag{6}$$

where the factor ig is for future convenience. Plugging the above formula into (5), we get:

$$D_\mu \psi = (\partial_\mu - igA_\mu)\psi \tag{7}$$

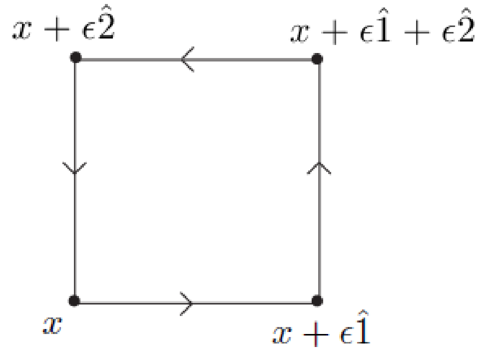

Figure 1: Wilson loop for an infinitesimal rectangle

Therefore, we successfully obtained the formula for the covariant derivative introduced in "Non-Abelian gauge theory!" Now, it is an easy exercise to derive the gauge transformation for A_μ. Inserting (6) into (3), we get:

$$\begin{aligned} 1 + ig\epsilon n^\mu A_\mu &\to V(x+\epsilon n^\mu)(1 + ig\epsilon n^\mu A_\mu)V^{-1}(x) + O(\epsilon^2) \\ 1 + ig\epsilon n^\mu A_\mu &\to (V(x) + \epsilon n^\mu \partial_\mu V(x))(1 + ig\epsilon n^\mu A_\mu)V^{-1}(x) + O(\epsilon^2) \\ ig\epsilon n^\mu A_\mu &\to \epsilon n^\mu \partial_\mu V(x)V^{-1}(x) + V(x) ig\epsilon n^\mu A_\mu V^{-1}(x) + O(\epsilon^2) \\ A_\mu(x) &\to VA_\mu(x)V^{-1}(x) - \frac{i}{g}(\partial_\mu V(x))V^{-1}(x) \end{aligned} \tag{8}$$

So, we recover our earlier formula in "Non-Abelian gauge theory!"

2 Wilson loop for an infinitesimal rectangle

$U(y, x)$ introduced in the last section is called the Wilson line. In this section, we introduce the Wilson loop, also called the "holonomy," which is a special case of the Wilson line. To this end, let's consider the case where $y = x$. Wait a minute. Haven't I just said that $U(x, x) = 1$? Indeed, but this case is a little bit different. See Fig. 1. In our earlier case, when I said $U(x, x) = 1$, there was no arrow, nor any path taken; it was meant to be the initial condition (i.e. boundary condition) for U. Now, in this case, $U(x, x)$ is given as follows:

$$\begin{aligned} U(x,x) = U(x, x+\epsilon\hat{2})&U(x+\epsilon\hat{2}, x+\epsilon\hat{1}+\epsilon\hat{2}) \\ &\times U(x+\epsilon\hat{1}+\epsilon\hat{2}, x+\epsilon\hat{1})U(x+\epsilon\hat{1}, x) \end{aligned} \tag{9}$$

I want to make three remarks about the above formula. First, the above formula is natural considering that $U(z, x) = U(z, y)U(y, x)$ from the definition of U and its gauge transformation (3). Second, the above formula depends on which path you take. In our case, the path was taken along the $\hat{1}$ and $\hat{2}$ directions. We would get different results, if we considered, say, the $\hat{2}$ and $\hat{3}$ directions. Third, the trace of the above formula (whether the path in question is infinitesimal or not) is gauge invariant, since

$$\mathrm{tr}U(x,x) \to \mathrm{tr}(V(x)U(x,x)V^{-1}(x)) = \mathrm{tr}U(x,x) \tag{10}$$

As an aside, I want to note that in Abelian gauge theory the Wilson loop is simply given by $U(x,x)$, since it is invariant under gauge transformation even without taking the trace.

Now, to gain more insight, let's explicitly calculate (9). To this end, we need to write (6) more precisely as follows:

$$U(x+\epsilon n, x) = \exp\left(ig\epsilon n^\mu A_\mu(x+\frac{\epsilon}{2}n) + O(\epsilon^3)\right) \tag{11}$$

where we have used the fact that U is unitary, which turns it into an exponential, and $(U(x,y))^\dagger = U(y,x)$, which leads us to use the mid-point of x and $x+\epsilon$ as the argument of A. This second condition makes the log of $U(z-\frac{\epsilon}{2}n, z+\frac{\epsilon}{2}n)$ an odd function of ϵ, which justifies the absence of a second order term in the above formula. Plugging (11) into (9), we get:

$$\begin{aligned}
U(x,x) = &\exp\left(-ig\epsilon A_2(x+\frac{\epsilon}{2}\hat{2}) + O(\epsilon^3)\right)\exp\left(-ig\epsilon A_1(x+\frac{\epsilon}{2}\hat{1}+\epsilon\hat{2}) + O(\epsilon^3)\right)\\
&\times \exp\left(+ig\epsilon A_2(x+\epsilon\hat{1}+\frac{\epsilon}{2}\hat{2}) + O(\epsilon^3)\right)\exp\left(+ig\epsilon A_1(x+\frac{\epsilon}{2}\hat{1}) + O(\epsilon^3)\right)\\
&= \left(1 - ig\epsilon A_2(x+\frac{\epsilon}{2}\hat{2}) - \frac{g^2\epsilon^2}{2}A_2^2 + O(\epsilon^3)\right)\left(1 - ig\epsilon A_1(x+\frac{\epsilon}{2}\hat{1}+\epsilon\hat{2}) - \frac{g^2\epsilon^2}{2}A_1^2 + O(\epsilon^3)\right)\\
&\times\left(1 + ig\epsilon A_2(x+\epsilon\hat{1}+\frac{\epsilon}{2}\hat{2}) - \frac{g^2\epsilon^2}{2}A_2^2 + O(\epsilon^3)\right)\left(1 + ig\epsilon A_1(x+\frac{\epsilon}{2}\hat{1}) - \frac{g^2\epsilon^2}{2}A_1^2 + O(\epsilon^3)\right)\\
&= 1 + i\epsilon^2 g[\partial_1 A_2 - \partial_2 A_1 - ig[A_1, A_2]] + O(\epsilon^3)\\
&= 1 + i\epsilon^2 g F_{12} + O(\epsilon^3)
\end{aligned} \tag{12}$$

So, we see that $U(x,x)$ is indeed gauge invariant. In other words, the Wilson loop contains gauge-invariant information about the connection. The Wilson loop is a true observable, unlike the connection itself which is gauge dependent. In fact, *Giles' theorem* says that we can reconstruct all the gauge-invariant information about the connection if we know all the possible Wilson loops in a given space. This fact plays an important role in loop quantum gravity, since it allows for a basis for the Hilbert space to be written in terms of the set of Wilson loops.

3 Differential equation for the Wilson line

Let's consider the following Wilson line. See Fig. 2. Here, s parametrizes the line γ^μ. For infinitesimal Δs we certainly have:

$$\begin{aligned}
U(\gamma(s+\Delta s), x) &= U(\gamma(s+\Delta s), \gamma(s))U(\gamma(s), x)\\
&= (1 + ig(\gamma^\mu(s+\Delta s) - \gamma^\mu(s))A_\mu(\gamma(s)))U(\gamma(s), x)
\end{aligned} \tag{13}$$

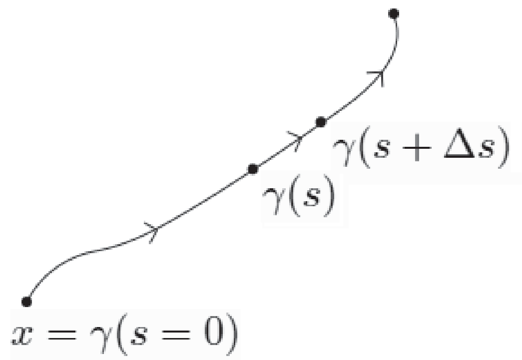

Figure 2: Wilson line

This implies:

$$\frac{d}{ds}U(\gamma(s), x) = ig\frac{d\gamma^\mu}{ds}A_\mu(\gamma(s))U(\gamma(s), x) \tag{14}$$

Now, let's check that the above formula transforms appropriately under gauge transformation. We have seen that the Wilson line transforms as follows:

$$U'(\gamma(s), x) = V(\gamma(s))U(\gamma(s), x)V^{-1}(x) \tag{15}$$

where $V(x) = e^{ig\theta^a(x)T^a}$. Differentiating, we have:

$$\begin{aligned}
\frac{dU'}{ds} &= \frac{d\gamma^\mu}{ds}\frac{\partial V(\gamma(s))}{\partial x^\mu}UV^{-1}(x) + V(\gamma(s))\frac{dU}{ds}V^{-1}(x) \\
&= \frac{d\gamma^\mu}{ds}[\frac{\partial V(\gamma(s))}{\partial x^\mu}UV^{-1}(x) + V(\gamma(s))igA_\mu UV^{-1}(x)] \\
&= \frac{d\gamma^\mu}{ds}[\frac{\partial V(\gamma(s))}{\partial x^\mu}V^{-1}(\gamma(s)) + V(\gamma(s))igA_\mu V^{-1}(\gamma(s))]U' \\
&= ig\frac{d\gamma^\mu}{ds}A'_\mu U'
\end{aligned} \tag{16}$$

where in the last line we used the gauge transformation law for A'_μ. So, the above equation is indeed gauge covariant. As an aside, it is also easy to see that (14) can be re-written in the following covariant form:

$$\frac{d\gamma^\mu}{ds}D_\mu U(\gamma(s), x) = 0 \tag{17}$$

We could have also said that this covariance made the covariance in (16) work.

4 Expressing the Wilson Loop using path ordering

What's the solution to the differential equation (14)? The solution is easy to obtain in the Abelian case. It is:

$$U(\gamma(s), x) = \exp\left(ig\int_0^s \dot{\gamma}^\mu(s')A_\mu(s')ds'\right) \tag{18}$$

where $\dot{\gamma}^\mu(s') \equiv d\gamma^\mu(s')/ds'$ is the tangent vector to the curve.

In the non-Abelian case, it is more difficult. Let's find this. (14) implies:

$$U(s) = 1 + ig \int_0^s ds_1 \dot{\gamma}^\mu(s_1) A_\mu(s_1) U(s_1) \tag{19}$$

where we have denoted $U(s) \equiv U(\gamma(s))$ and $A_\mu(s) \equiv A_\mu(\gamma(s))$. Now, inserting the left-hand side (i.e. $U(s)$ into the right-hand side (i.e. $U(s_1)$), we obtain:

$$\begin{aligned} U(s) = 1 &+ ig \int_0^s ds_1 \dot{\gamma}^\mu(s_1) A_\mu(s_1) \\ &- g^2 \int_0^{s_1} ds_1 \dot{\gamma}^\mu(s_1) A_\mu(s_1) \int_0^{s_1} ds_2 \dot{\gamma}^\mu(s_2) A_\mu(s_2) U(s_2) \end{aligned} \tag{20}$$

Now inserting (19) into $U(s_2)$ again, we obtain:

$$\begin{aligned} U(s) = 1 &+ ig \int_0^s ds_1 \dot{\gamma}^\mu(s_1) A_\mu(s_1) \\ &- g^2 \int_0^{s_1} ds_1 \dot{\gamma}^\mu(s_1) A_\mu(s_1) \int_0^{s_1} ds_2 \dot{\gamma}^\mu(s_2) A_\mu(s_2) \\ &- ig^3 \int_0^{s_1} ds_1 \dot{\gamma}^\mu(s_1) A_\mu(s_1) \int_0^{s_1} ds_2 \dot{\gamma}^\mu(s_2) A_\mu(s_2) \int_0^{s_2} ds_3 \dot{\gamma}^\mu(s_3) A_\mu(s_3) U(s_3) \end{aligned}$$

Repeating the process, we obtain:

$$U(s) = \sum_{n=0}^{\infty} \left[(ig)^n \int_{s \geq s_1 \geq \cdots \geq s_n \geq 0} ds_1 \cdots ds_n \dot{\gamma}^{\mu_1}(s_1) A_{\mu_1}(s_1) \cdots \dot{\gamma}^{\mu_n}(s_n) A_{\mu_n}(s_n) \right] \tag{21}$$

Actually, we can express the above formula more succinctly. To this end, note that the various factors of $\dot{\gamma}^\mu A_\mu$ stand in "path order." That is, the larger values of the s's stand to the left. From this observation, we introduce the "path ordered product:"

$$P(A_{\mu_1}(s_1) \cdots A_{\mu_n}(s_n)) \tag{22}$$

which rearranges the products in such a way that the larger values of the s's stand to the left. For example, if $s_3 > s_1 > s_4 > s_2$

$$P(A_{\mu_1}(s_1) A_{\mu_2}(s_2) A_{\mu_3}(s_3) A_{\mu_4}(s_4)) = A_{\mu_3}(s_3) A_{\mu_1}(s_1) A_{\mu_4}(s_4) A_{\mu_2}(s_2) \tag{23}$$

Now, convince yourself of the following:

$$\int_{s \geq s_1 \geq s_2} ds_1 ds_2 A_{\mu_1}(s_1) A_{\mu_2}(s_2) = \frac{1}{2} \int_0^s ds_1 \int_0^s ds_2 P(A_{\mu_1}(s_1) A_{\mu_2}(s_2)) \tag{24}$$

This is so since there are two cases which contribute equally to the integration, given the condition $s \geq s_1 \geq 0$ and $s \geq s_2 \geq 0$. Namely, $s_1 \geq s_2$ and $s_2 \geq s_1$.

Similarly one can show

$$\begin{aligned} & \int_{s \geq s_1 \geq \cdots \geq s_n \geq 0} ds_1 \cdots ds_n \dot{\gamma}^{\mu_1}(s_1) A_{\mu_1}(s_1) \cdots \dot{\gamma}^{\mu_n}(s_n) A_{\mu_n}(s_n) \\ = \; & \frac{1}{n!} \int_0^s ds_1 \cdots ds_n P(\dot{\gamma}^{\mu_1}(s_1) A_{\mu_1}(s_1) \cdots \dot{\gamma}^{\mu_n}(s_n) A_{\mu_n}(s_n)) \\ = \; & \frac{1}{n!} P\left[\left(\int_0^s ds' \dot{\gamma}^\mu(s') A_\mu(s') \right)^n \right] \end{aligned} \tag{25}$$

Convince yourself again that this is correct. For example, when $n = 3$, we have to divide by 3!, as there are 3! cases that contribute equally. Namely:

$$\begin{aligned} &s_1 \geq s_2 \geq s_3, \quad s_1 \geq s_3 \geq s_2 \\ &s_2 \geq s_3 \geq s_1, \quad s_2 \geq s_1 \geq s_3 \\ &s_3 \geq s_1 \geq s_2, \quad s_3 \geq s_2 \geq s_1 \end{aligned}$$

Finally, we define the path ordered exponential:

$$P\left[\exp\left(ig\int_0^s ds'\dot{\gamma}^\mu(s')A_\mu(s')\right)\right] \equiv \sum_{n=0}^{\infty}\frac{(ig)^n}{n!}P\left[\left(\int_0^s ds'\dot{\gamma}^\mu(s')A_\mu(s')\right)^n\right] \tag{26}$$

This last line is the Wilson loop $U(s)$ expressed succintly in a form that closely resembles the Abelian case given in (18).

Problem 1. Express (18) in terms of $F_{\mu\nu}$ using Green's theorem. Then, notice that the result is valid even in non-Abelian case by comparing with (12).

Summary

- A Wilson line is given by
$$U \sim P[\exp(ig\int A)]$$
where P is the path ordered exponential.

- A Wilson loop is a Wilson line that closes itself.

- An infinitesimal Wilson loop can be expressed as
$$U \sim 1 + iaF_{12}$$
if the loop lies in 1-2 plane. Here, a is proportional to the area of the loop.