Polynomials, expansion and factoring

A polynomial is an expression that can be written as the sum of $ax_1^{n1}x_2^{n_2}x_3^{n_3}\cdots x_m^{n_m}$ where a is a number, xs are variables, and ns are non-negative integers. For example, the following expressions are all polynomials:

$$xy^3, \qquad 2y^2z - 3xyz - x, \qquad 3x^2yz + 4y^2z, \qquad 6xyz^2 + 5x + 3$$

The degree of a term is the sum of the exponents in a term. For example, the degree of xy^3 is 4 as $xy^3 = x^1y^3$ and $1 + 3 = 4$. Similarly, the degree of $-3x^4yz$ is $4 + 1 + 1 = 6$.

The degree of a polynomial is the largest of the degrees of each term. For example, $2xy^2z - 4wxyz^3$ has a degree 6 as $2xy^2z$ has a degree 4 and $-4wxyz^3$ has a degree 6. Expressions which have degree 1 are called "linear." For example $x + y = 4$ and $x - y = -3$ are considered as linear equations.

Polynomials can be added, subtracted, and multiplied. The result of this are still polynomials. For example,

$$(xy + 4xy^2) + (-xy + 3xy^2) = 7xy^2 \tag{1}$$

$$(6y + 7xyz) - (5y - 3xyz) = y + 10xyz \tag{2}$$

$$(a + b)(c + d) = (a + b)c + (a + b)d = ac + bc + ad + bd \tag{3}$$

where in the last equation we used the distributive property.

However, when a polynomial is divided by another polynomial, the result can be a non-polynomial. For example, the following is not polynomial,

$$(xy + yz) \div y^2 = xy/y^2 + yz/y^2 = \frac{x}{y} + \frac{z}{y} \tag{4}$$

as the above expression has negative exponents (i.e. $xy^{-1} + zy^{-1}$)

Now comes an important concept. An expansion of a product of sums expresses it as a sum of products by using the distributive property. In other words, you remove the parenthesis. (3) is a good example. Here comes some important expansions.

$$\begin{aligned}(a + b)^2 &= (a + b)(a + b)\\ &= (a + b)a + (a + b)b\\ &= a^2 + ba + ab + b^2\\ &= a^2 + 2ab + b^2 \qquad (5)\end{aligned}$$

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b) \\ &= (a-b)a - b(a-b) \\ &= a^2 - ba - ab + b^2 \\ &= a^2 - 2ab + b^2 \end{aligned} \tag{6}$$

$$\begin{aligned}(a+b)(a-b) &= (a+b)a - b(a+b) \\ &= a^2 + ab - ba - b^2 \\ &= a^2 - b^2 \end{aligned} \tag{7}$$

I recommend you memorize these three formulas. Using the distributive property iteratively, one can expand more complicated expressions. For example,

$$\begin{aligned}(a+b)^3 &= (a+b)(a+b)^2 \\ &= (a+b)(a^2+2ab+b^2) \\ &= a(a^2+2ab+b^2) + b(a^2+2ab+b^2) \\ &= a^3 + 2a^2b + ab^2 + a^2b + 2ab^2 + b^3 \\ &= a^3 + 3a^2b + 3ab^2 + b^3 \end{aligned} \tag{8}$$

On the other hand, factoring is the reverse process of expansion. For example,

$$2x + 4y = 2(x+2y), \qquad x^2 - 4y^2 = (x+2y)(x-2y), \qquad x^2 + 8x + 16 = (x+4)^2$$

Problem 1. Expand the following.

$$(a-b)^3 = ?, \qquad (a+b)(a^2-ab+b^2) = ?, \qquad (a-b)(a^2+ab+b^2) = ?$$

$$(x+3)(x+2) = ?, \qquad (3x+2)(2x-3) = ?, \qquad (3x^2+1)(x-2) = ?$$

Problem 2. Factor out the following.

$$2xy + 4xyz = ?, \qquad x^2 - 4 = ?, \qquad x^2 + 2x + 1 = ?, \qquad 3x - 9xy = ?$$

Problem 3. Simplify the following. (We assume $a \neq b$. Hint[1])

$$\frac{a-b}{\sqrt{a}-\sqrt{b}} = ?, \qquad \frac{1}{x-1} - \frac{1}{x-2} = ?$$

Problem 4. Prove the following. These relations are useful in our article "Quadratic equation."

$$\sqrt{\left(\frac{b}{2a}\right)^2 - \frac{c}{a}} = \frac{\sqrt{b^2-4ac}}{2a}, \qquad (x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta$$

Problem 5. Prove the following. This is useful in string theory. (We assume $x \neq 0, 1$. Hint[2])

$$x^u(1-x)^{v-1} + x^{u-1}(1-x)^v = x^{u-1}(1-x)^{v-1} \tag{9}$$

[1] Use $(a-b) = (\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b})$ and $\frac{1}{c} - \frac{1}{d} = \frac{d}{cd} - \frac{c}{cd} = \frac{d-c}{cd}$.
[2] Use $x^u = x^{u-1}x$ and $(1-x)^v = (1-x)^{v-1}(1-x)$.

Summary

- A polynomial is an expression that can be written as the sum of $ax_1^{n1}x_2^{n_2}x_3^{n_3}\cdots x_m^{n_m}$ where a is a number, xs are variables, and ns are non-negative integers.
- The degree of a term is the sum of the exponents in a term.
- The degree of a polynomial is the largest of the degrees of each term.
- When polynomials are added, subtracted, or multiplied, the result of this are still polynomials.
- However, when a polynomial is divided by another polynomial, the result can be a non-polynomial.
- An expansion of a product of sums expresses it as a sum of products by using the distributive property. In other words, you remove the parenthesis.
- $(a+b)^2 = a^2 + 2ab + b^2$, $\qquad (a-b)^2 = a^2 - 2ab + b^2$, $\qquad (a+b)(a-b) = a^2 - b^2$
- Factoring is the reverse process of expansion.