Polynomial division

In the last article, we have seen that in the case in which a polynomial is divided by another polynomial the result can be non-polynomial. This is similar to the division of integers. When an integer is divided by another integer, the answer can be a non-integer. However, remember that there is another way to divide an integer by another integer: when you divide 7 by 3 the quotient is 2 and the remainder is 1. This is so as $7 = 3 \times 2 + 1$. Similarly, we can divide a polynomial by another polynomial to get a quotient and a remainder. It turns out that this can be done consistently for polynomials of a single variable. For example, $x^3 + 2x^2 + 4$ is a polynomial of a single variable x while $xy^2 + xy + z$ is a polynomial of three variables x, y and z. Given this, let me provide an example of dividing a polynomial of x by another polynomial of x. If you divide $x^5 + 3x^3 + x + 4$ by $x^2 + 1$, the quotient is $x^3 + 2$ and the remainder is $x + 2$. This is so as $x^5 + 3x^3 + x + 4 = (x^2 + 1)(x^3 + 2) + x + 2$. Remember that in our earlier example of the division of integers, the remainder 1 must be always smaller than the divisor 3. Similarly, in the case of the division of polynomials, the degree of the remainder must be always smaller than the degree of the divisor. In our case, the degree of $x + 2$ is 1 and the degree of $x^2 + 1$ is 2. We see that 1 is indeed smaller than 2.

Then, how can we easily perform the division of polynomials? The trick is similar to the trick for integers. Let's divide 1740 by 8. We have:

$$
\begin{array}{r}
217 \\
8\overline{)1740} \\
16 \\
\overline{14} \\
8 \\
\overline{60} \\
56 \\
\overline{4}
\end{array}
$$

Therefore, the quotient is 217 and the remainder is 4.

Similarly, if we divide $x^5 + x^4 - x^2 + 4x - 2$ by $x^2 + 2x + 3$, we have:

$$
\begin{array}{r}
x^3 - x^2 - x + 4 \\
x^2 + 2x + 3 \overline{) x^5 + x^4 + 0x^3 - x^2 + 4x - 2} \\
\underline{x^5 + 2x^4 + 3x^3} \qquad\qquad\qquad \\
- x^4 - 3x^3 - x^2 \qquad\qquad \\
\underline{-x^4 - 2x^3 - 3x^2} \qquad\qquad \\
- x^3 + 2x^2 + 4x \qquad \\
\underline{-x^3 - 2x^2 - 3x} \qquad \\
4x^2 + 7x - 2 \\
\underline{4x^2 + 8x + 12} \\
- x - 14
\end{array}
$$

The quotient is $x^3 - x^2 - x + 4$ and the remainder is $-x - 14$.

Problem 1. Divide $x^3 + x^2 - 1$ by $x + 2$ to find the quotient and the remainder.

Summary

- We can divide a polynomial by another polynomial to get a quotient and a remainder.
- The degree of the remainder must be always smaller than the degree of the divisor.