Quadratic equation

You know how to solve the following type of equations.

$$x^2 = 2, \qquad y^2 = 4, \qquad z^2 = 81 \tag{1}$$

The answers are

$$x = \pm\sqrt{2}, \qquad y = \pm 2, \qquad z = \pm 9 \tag{2}$$

where $\pm$ here denotes the fact that the answer can take both positive and negative values. For example, $\pm 2 = 2, -2$.

However, the following type of equations are much more tricky:

$$x^2 + 6x + 5 = 0 \tag{3}$$

In this article, I will teach you how to solve this type of equations, which is called a "quadratic equation." A quadratic equation is in the following form.

$$ax^2 + bx + c = 0 \tag{4}$$

where a, b, and c are constants.

Now, back to (3). Let's solve it.

$$\begin{aligned} (x^2 + 6x + 9 - 9) + 5 &= 0 \\ (x+3)^2 - 9 + 5 &= 0 \\ (x+3)^2 &= 4 \\ x + 3 &= 2, -2 \\ x &= -1, -5 \end{aligned} \tag{5}$$

So, we solved it! Note the crucial step. You factor out x^2+6x+9 by $(x+3)^2$. Remember, $(x+d)^2 = x^2 + 2dx + d^2$. In our case we set $d = 3$ because $2dx$ must match $6x$ in (3). So, we set $x^2 + 2dx = (x+d)^2 - d^2$. This procedure is known as "completing the square."

One more example:

$$\begin{aligned} x^2 - 4x - 3 &= 0 \\ x^2 - 4x + 4 - 4 - 3 &= 0 \\ (x+2)^2 - 7 &= 0 \\ (x+2)^2 &= 7 \\ x + 2 &= \pm\sqrt{7} \\ x &= -2 \pm \sqrt{7} \end{aligned} \tag{6}$$

What if a in (4) is not 1? If we simply divide both sides by a, the coefficient in front of x^2 will be 1. Then, we can just take similar steps as before. For example,

$$
\begin{aligned}
6x^2 - 7x - 3 &= 0 \qquad (7)\\
x^2 - \frac{7}{6}x - \frac{1}{2} &= 0\\
(x - \frac{7}{12})^2 - \frac{49}{144} - \frac{1}{2} &= 0\\
(x - \frac{7}{12})^2 &= \frac{121}{144}\\
x - \frac{7}{12} &= \frac{11}{12}, -\frac{11}{12}\\
x &= \frac{3}{2}, -\frac{1}{3} \qquad (8)
\end{aligned}
$$

Let me just present one more example:

$$
\begin{aligned}
4x^2 + 5x - 3 &= 0 \qquad (9)\\
x^2 + \frac{5}{4}x - \frac{3}{4} &= 0\\
(x + \frac{5}{8})^2 - \frac{25}{64} - \frac{3}{4} &= 0\\
(x + \frac{5}{8})^2 &= \frac{73}{64}\\
x + \frac{5}{8} &= \pm\frac{\sqrt{73}}{8}\\
x &= \frac{-5 \pm \sqrt{73}}{8} \qquad (10)
\end{aligned}
$$

So far, we have seen the cases in which there are two solutions. However, it is possible for there to be only one solution or no solutions at all. For example, as you see, the following equation has only one solution.

$$
\begin{aligned}
x^2 - 4x + 4 &= 0\\
(x - 2)^2 &= 0\\
x - 2 &= 0\\
x = 2 & \qquad (11)
\end{aligned}
$$

Notice that in the earlier cases there are two solutions because the equation of type $(x + d)^2 = e$ implies $x + d = \pm\sqrt{e}$. This means that we had two cases, the plus sign and the negative sign, as long as e is positive. However, in the present case ± 0 is not two numbers, but only one number, namely 0.

Furthermore, it sometimes happens that there is no solution. For example, $x^2 = -1$ has no solution. If you square a number, it is always non-negative. Similarly, the following equation has no solution,

$$x^2 + 2x + 2 = 0 \quad (12)$$
$$(x+1)^2 = -1 \quad (13)$$

since no number squared can be a negative number. In other words, we made the equation into a form $(x+d)^2 = e$ when e is negative. As an aside, in our later article on complex numbers, we will demonstrate a way to provide solutions for these kind of equations, by defining a number that becomes a negative number when squared. But this number is not "real," so we call it "imaginary." At present, just accept that there is no such ordinary (i.e. real) number.

Let me also mention that if α is a solution to $x^2 + bx + c = 0$, it can be factorized as follows:

$$x^2 + bx + c = (x-\alpha)(x-\beta) \quad (14)$$

This can be seen below. First, notice that $x^2 + bx + c$ can be always expressed as follows

$$x^2 + bx + c = (x-\alpha)(x-\beta) + \gamma \quad (15)$$

for some β and γ. (If you divide $x^2 + bx + c$ by $(x-\alpha)$, then $(x-\beta)$ is the quotient and γ is the remainder.) Now, let's plug in $x = \alpha$ to both sides. The left-hand side is zero, while the right-hand side is γ. So, we conclude $\gamma = 0$. This brings us back to (14). We also see that β is the other solution by plugging $x = \beta$ to both sides. Moreover, (14) allows us to express b and c in terms of the solution α and β. By expanding, we get:

$$x^2 + bx + c = x^2 - (\alpha+\beta)x + \alpha\beta \quad (16)$$

Therefore, we conclude:

$$b = -(\alpha+\beta), \qquad c = \alpha\beta \quad (17)$$

Let me conclude this article with a comment. We can solve any quadratic equations using the method presented in this article. However, this trick doesn't work for other types of equations such as cubic or quartic equations. Cubic equations can be represented in the form $ax^3 + bx^2 + cx + d = 0$. When I was still a young child, I read from a cartoon book that the solution to any cubic equation was found in the 16th century. So, I told my mom that I wanted to know the solution. She took me to the biggest library in Daejeon and found out a book that described this. It was before the time when the Internet was widely available and before the time when libraries used computers to aid in searching for a book. I also guessed that the solutions to cubic equations were important, since the solutions to quadratic equations are important, and cubic equations seemed to be the next least complicated equations. However, I came to learn much later that the solutions to the cubic equations are much less important in everyday lives or in physics or engineering than I imagined. Therefore, most university students, regardless of their majors do not learn the solution to the cubic equations even though it is not that difficult.

The solution to quartic equations (i.e. equations of the form $ax^4+bx^3+cx^2+dx+e=0$) was also found in the 16th century, but the solutions to a higher degree of equations (such as $ax^5+bx^4+cx^3+dx^2+ex+f=0$) were never found. However, the 19th century French mathematician Évariste Galois proved that a general solution to such equations cannot exist. In other words, one cannot solve such equations using addition, subtraction, multiplication, division and roots only. (Of course, one can still obtain the solution numerically using computers.) The proof uses Galois theory, which math majors learn in their sophomore years. At present, Galois theory is a branch of mathematics that finds no applications in physics or engineering. Nevertheless, I read a Korean translation of a book on Galois theory aimed at laymen written by a Korean Japanese translator and novelist. The title roughly translates into English as "Galois theory which I teach to my thirteen-year old daughter." It is a shame that this book is not translated yet into English.

Problem 1. Solve the following equations. (Hint[1])

$$(4x+3)(x-3)=2x-16, \qquad (x+3)(x-2)=x^2+4x+6$$

Problem 2. If the solutions to $x^2+bx+c=0$ are 2 and 3, what are b and c?

Problem 3. If the solutions to $6x^2+dx+e=0$ are $\frac{1}{2}$ and $\frac{1}{3}$, what are d and e? (Hint[2])

Problem 4. Solve the following equation. ($a \neq 0$, Hint[3])

$$ax^2+bx+c=0$$

If you correctly solve, you will get

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \tag{18}$$

$D=b^2-4ac$ is called a "discriminant." How is the sign of the discriminant related to the number of solutions? More specifically, when do we have two solutions? one solution? no solution? (Hint[4])

[1] First, make them into the form (4).

[2] Notice $6(x^2+\frac{d}{6}x+\frac{e}{6})=0$.

[3] Take the same steps as the one that led (7) to (8) and the one that led (9) to (10). The first part of Problem 4 in "Polynomials, expansion and factoring" is helpful.

[4] Read carefully again our discussions after (11) and (13).

Problem 5. Solve the following equations. (Hint[5])

$$\frac{1}{x-1}+\frac{1}{x-2}=\frac{3}{2}, \qquad \sqrt{x^2-3x}=x+2 \tag{19}$$

Problem 6. Solve the following equations. (Hint[6])

$$x+y=1 \tag{20}$$
$$x^2=4y/3 \tag{21}$$

Problem 7. If the following equation is an identity, what are the values for a, b, and c?

$$ax^2+3x+4=2x^2+bx+c \tag{22}$$

Problem 8. If the following equation is an identity, what are the values for a, b, and c?

$$x+3=ax^2+bx(x+1)+c(x+1) \tag{23}$$

Problem 9. If the following equation is an identity, what are the values for a, b, and c? (We assume $x \neq -1, 0$. Hint[7])

$$\frac{x+\frac{1}{2}}{x(x+1)^2}=\frac{a}{x}+\frac{b}{x+1}+\frac{c}{(x+1)^2} \tag{24}$$

Summary

- A type of equations that can be reduced to $ax^2+bx+c=0$ for $a\neq 0$, is called a "quadratic equation."
- A quadratic equation can be solved by completing the square.
- $(x+d)^2=e$ has two solutions if $e>0$, one solution $e=0$, no solution $e<0$.
- If α and β are solutions to $x^2+bx+c=0$, we necessarily have

$$x^2+bx+c=(x-\alpha)(x-\beta)$$

- The solution to $ax^2+bx+c=0$ where $a\neq 0$, is given by

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

  $D=b^2-4ac$ is called a "discriminant." If $D>0$ there are two solutions, if $D=0$ there are one solution, if $D<0$ there is no solution.

[5] For the first one, multiply both sides by $2(x-1)(x-2)$. For the second one, notice that the equation implies $(x^2-3x)=(x+2)^2$.

[6] Show first that the equations imply $x+3x^2/4=1$.

[7] Multiply both sides by $x(x+1)^2$.