

## Quadratic inequalities and the Cauchy-Schwarz inequality

Having solved quadratic equations, we can now solve quadratic inequalities. Suppose we want to solve the following inequality. We will assume  $a > 0$ , as we can consider the case where  $a < 0$  later.

$$ax^2 + bx + c > 0 \tag{1}$$

Let's draw a graph for this inequality. See the figures on the next page. There are three cases. In the first case, there is no solution to  $ax^2 + bx + c = 0$ , as the minimum of  $ax^2 + bx + c$  is bigger than zero. In this case all  $x$  satisfy (1). If you correctly solved the problem in our earlier article, you will also see that this is the case when the discriminant is less than zero.

In the second case, there is one solution to  $ax^2 + bx + c = 0$ . If you correctly solved the problem in our earlier article, you will see that this corresponds to the case in which the discriminant is zero. Since  $ax^2 + bx + c$  is always positive except when it is zero, the answer to (1) is given by all  $x \neq \alpha$  where  $\alpha$  is the (only) solution to  $ax^2 + bx + c = 0$ .

**(Problem 1.** In this second case, using the fact that  $x = \alpha$  is the only solution to  $ax^2 + bx + c = 0$ , express  $\alpha$  in terms of  $a$  and  $b$ .)

In the third case, there are two solutions to  $ax^2 + bx + c = 0$ . If the smaller solution is  $\alpha$  while the bigger solution is  $\beta$ , it is obvious from the graph that  $ax^2 + bx + c$  is positive when  $x < \alpha$  or  $x > \beta$ . This is the answer.

**Problem 2.** Solve  $ax^2 + bx + c \geq 0$  for  $a > 0$ . (Hint<sup>1</sup>)

**Problem 3.** Solve  $ax^2 + bx + c < 0$  for  $a > 0$ . (Hint<sup>2</sup>) Notice also that if  $a < 0$  in (1), it can be converted to this type of inequality. For example, if we are to solve  $-2x^2 + 3x + 4 > 0$ , we can simply solve instead  $0 > 2x^2 - 3x - 4$ .

**Problem 4.** Notice that the following is satisfied for all  $t$ , as it is a sum of two non-negative numbers (if you square something you always get something non-negative):

$$(tX_1 + X_2)^2 + (tY_1 + Y_2)^2 \geq 0 \tag{2}$$

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<sup>1</sup> $ax^2 + bx + c \geq 0$  is satisfied whenever  $ax^2 + bx + c > 0$  or  $ax^2 + bx + c = 0$ .

<sup>2</sup>Except for when  $ax^2 + bx + c > 0$  or  $ax^2 + bx + c = 0$  is satisfied, this inequality is always satisfied. Either use this fact or think on your own by looking at the three figures closely again.

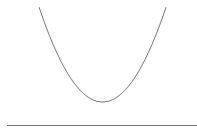


Figure 1:  $b^2 - 4ac < 0$

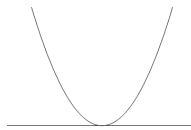


Figure 2:  $b^2 - 4ac = 0$

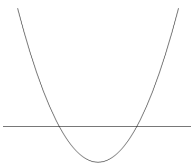


Figure 3:  $b^2 - 4ac > 0$

If you expand this, you get a degree-2 polynomial of  $t$ . By using the fact that its discriminant must be zero or negative, show the following:

$$(X_1X_2 + Y_1Y_2)^2 \leq (X_1^2 + Y_1^2)(X_2^2 + Y_2^2) \quad (3)$$

This is known as the Cauchy-Schwartz inequality. More generally, the following is satisfied, if we consider  $(tX_1 + X_2)^2 + (tY_1 + Y_2)^2 + (tZ_1 + Z_2)^2 + \dots \geq 0$ :

$$(X_1X_2 + Y_1Y_2 + Z_1Z_2 + \dots)^2 \leq (X_1^2 + Y_1^2 + Z_1^2 + \dots)(X_2^2 + Y_2^2 + Z_2^2 + \dots) \quad (4)$$

We will use this formula to prove the “triangle inequality” later.

Final comment. In our later article on quantum mechanics, you will learn Heisenberg’s uncertainty principle. According to this principle, you cannot measure position and velocity of a particle both accurately at the same time. The proof of Heisenberg’s uncertainty principle uses the Cauchy-Schwarz inequality.

**Problem 5.** Solve  $3x < x^2 + 2$ . (Hint<sup>3</sup>)

**Problem 6.** Solve  $x^2 < 6x - 10$ . (Hint<sup>4</sup>)

**Problem 7.** Can the square of a certain number ever be smaller than the original number? At first glance, it may seem unlikely. If you square 2, it is 4 which is not smaller than 2. If you square 100, it is 10000 which is not smaller than 100. The situation does not change for negative numbers. If you square  $-3$  it is 9, which is not smaller than  $-3$ . However, show that the square of a certain number *can be smaller* than the original number, by solving  $x^2 < x$ . After obtaining the solution, check yourself with an explicit example that the square can be smaller than the original number.

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<sup>3</sup>Solve  $x^2 - 3x + 2 > 0$

<sup>4</sup>Solve  $x^2 - 6x + 10 < 0$

## Summary

- The Cauchy-Schwartz inequality can be obtained by considering the discriminant of a quadratic inequality and is given by

$$(X_1X_2 + Y_1Y_2)^2 \leq (X_1^2 + Y_1^2)(X_2^2 + Y_2^2)$$