A short introduction to quantum mechanics XII: Heisenberg's uncertainty principle

Heisenberg's uncertainty principle asserts that the more precisely you know an object's position, the less precisely you know its momentum and vice versa. This fact can be derived rigorously by mathematics. In this article, we explain Heisenberg's uncertainty principle and mathematically derive it.

Suppose an object traveling with a certain, exact, velocity in the positive x-direction. In other words, it is traveling with a certain, exact momentum. This would imply that the wave function of the object is the eigenstate of the momentum operator. If you recall what we have discussed in our last article, you will see that the wave function in this case should be given as follows:

$$\psi = C(\cos(ipx/\hbar) + i\sin(ipx/\hbar)) \tag{1}$$

Let's draw this wave function. See Fig. 1. For simplicity, we have only drawn the imaginary part of the wave function. So, the momentum of the object has a certain, exact value. On the other hand, it would be hard to locate its position. Where is it located? At $x = -6$? $x = -3$? $x = 0$? $x = 4$? There is no fixed location, and it has actually equal probability to be found anywhere! As just stated in the beginning of the article, as we know the object's momentum very precisely, we cannot locate its position precisely. Heisenberg's uncertainty principle is mathematically stated as follows:

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}, \quad \Delta y \Delta p_y \geq \frac{\hbar}{2}, \quad \Delta z \Delta p_z \geq \frac{\hbar}{2}, \quad \Delta E \Delta t \geq \frac{\hbar}{2} \tag{2}$$

where Δx is the standard deviation of the x-position, and Δp_x is the standard deviation of the x-momentum and so on.

In our case for Fig. 1. $\Delta p_x = 0$, which forces $\Delta x = \infty$ from above equation. On the other hand, see Fig. 2, which is somewhat the opposite case of Fig. 1. The object is well-located at $x = 0$ with Δx roughly being around 1. However, the value for the wavelength cannot be quite well-determined, since there are only $3 \sim 4$ oscillations. So, the standard deviation of the wavelength is big. (After all, if it were zero, we would have had Fig. 1.) Since the wavelength gives the value for the momentum by the de-Broglie

Fig. 1

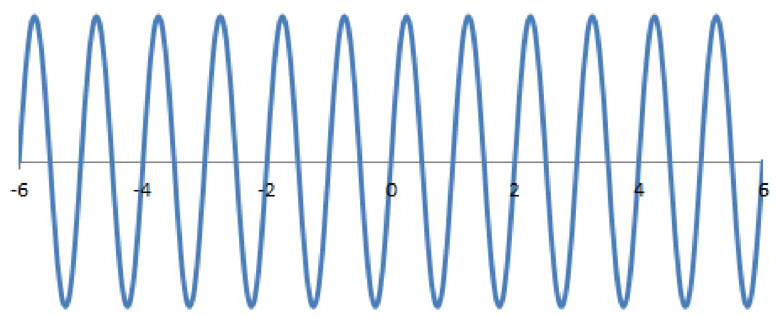


Fig. 2

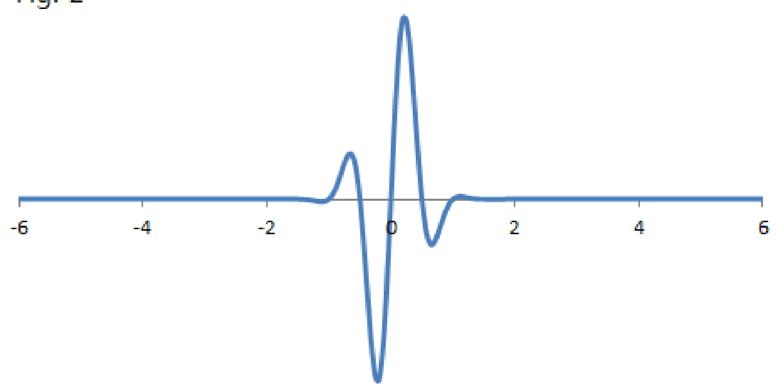
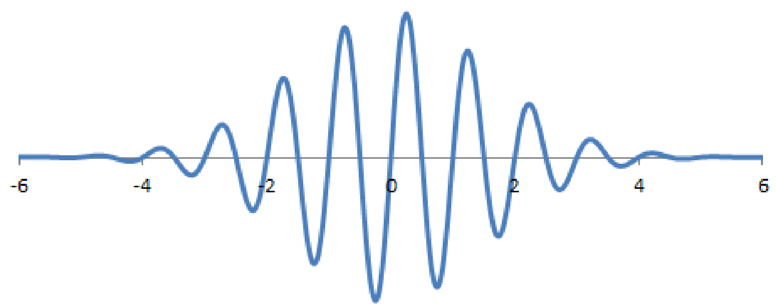


Fig. 3



formula, ($p = h/\lambda$) we can say that the standard deviation of the momentum is large. Again, we recover our assertion that the more precisely you know an object's location, the less precisely you know its momentum.

For just one more example, we have our third case Fig. 3. Its Δx is smaller than that of Fig. 1 while bigger than that of Fig. 3. Therefore, according to (2), it is allowed that Fig. 2's Δp_x is bigger than that of Fig. 1 while smaller than that of Fig. 3. This is actually the case. Fig. 3 looks closer to Fig. 1 than Fig. 2 does. Therefore, Fig. 2's Δx and Δp_x are between those of Fig. 1 and Fig. 2.

Now, let's derive Heisenberg's uncertainty principle. Consider two Hermitian operators F,G. We certainly have:

$$FG = \frac{1}{2}(FG + GF) - \frac{i}{2}i(FG - GF) \tag{3}$$

Taking the expectation value, we have:

$$< FG >= \frac{1}{2} < FG + GF > - \frac{i}{2} < i(FG - GF) > \tag{4}$$

Given this, notice followings:

$$(FG + GF)^\dagger = F^\dagger G^\dagger + G^\dagger F^\dagger = FG + GF \tag{5}$$

$$(iFG - iGF)^\dagger = iF^\dagger G^\dagger - iG^\dagger F^\dagger = iFG - iGF \tag{6}$$

Therefore, the first term in (3) is real, while the second term is purely imaginary. This leads to the following:

$$\begin{aligned} | < FG > |^2 &= \frac{1}{4} < FG + GF >^2 + \frac{1}{4} < FG - GF >^2 \\ | < FG > |^2 &\geq \frac{1}{4} < [F, G] >^2 \end{aligned} \tag{7}$$

Now, let

$$F = A- < A >, \quad G = B- < B > \tag{8}$$

since $< A >$ and $< B >$ are mere numbers, we have:

$$[F, G] = [A, B] \tag{9}$$

Plugging (8) and (9) to (7), we obtain:

$$|(< A- < A >)(B- < B >) > |^2 \geq \frac{1}{4} < [A, B] >^2 \tag{10}$$

On the other hand, from Schwartz's inequality (if you don't know what it is, read our earlier article "Quadratic inequalities and the Cauchy-Schwarz inequality." There the derivation was done in real vector space, but one can easily derive its complex vector space version, by being careful of the fact that $|v>= a|w>$ implies $<v| = a*|w>$), we have:

$$\begin{aligned} <A-<A>>^2<B-<B>>^2 &\geq |<(A-<A>)(B-<B>)>|^2 \\ (\Delta A)^2(\Delta B)^2 &\geq |<(A-<A>)(B-<B>)>|^2 \\ (\Delta A)^2(\Delta B)^2 &\geq \frac{1}{4}<[A,B]>^2 \\ (\Delta A)(\Delta B) &\geq \frac{1}{2}<[A,B]> \end{aligned} \tag{11}$$

where from the second line to the third line, we used (10).

If we plug in $A = x$, $B = p_x$, we obtain:

$$\Delta x \Delta p_x \geq \frac{\hbar}{2} \tag{12}$$

And similarly for others. However, we cannot prove $\Delta E \Delta t \geq \hbar/2$ using this method; t is just a coordinate, not a Hermitian operator. Nevertheless, it sounds reasonable, if you look at this from the point of view of Fourier transformation. Let me clarify what I mean. If we write

$$\psi(x, y, z, t) = \int \frac{dE dp_x dp_y dp_z}{(2\pi\hbar)^2} \phi(E, p_x, p_y, p_z) e^{i(p_x x + p_y y + p_z z - Et)/\hbar} \tag{13}$$

then, Heisenberg's uncertainty principle says that there are following relations between ψ and ϕ.

$$\left(\int x^2 \psi\psi^* dx - \left(\int x\psi\psi^* dx \right)^2 \right) \left(\int p_x^2 \phi\phi^* dp_x - \left(\int p_x \phi\phi^* dp_x \right)^2 \right) \geq \frac{\hbar^2}{4}$$

In other words, this is a relation that says about a property of Fourier transformation. As this property must be satisfied for other conjugate variables for Fourier transformation, we can write

$$\left(\int t^2 \psi\psi^* dt - \left(\int t\psi\psi^* dt \right)^2 \right) \left(\int E^2 \phi\phi^* dE - \left(\int E \phi\phi^* dE \right)^2 \right) \geq \frac{\hbar^2}{4}$$

which implies $\Delta E \Delta t \geq \hbar/2$.

Finally, notice also that plugging $A = y$, $B = p_z$ into (11) yields:

$$\Delta y \Delta p_z \geq 0 \tag{14}$$

Therefore, if the component measured for the position and the one for the momentum are different, there is no restriction for uncertainty; knowing the y-position of an object doesn't hinder from knowing its z-momentum precisely.

Summary

- Heisenberg's uncertainty relation is given by

$$\Delta x \Delta p_x \geq \frac{\hbar}{2}, \quad \Delta y \Delta p_y \geq \frac{\hbar}{2}, \quad \Delta z \Delta p_z \geq \frac{\hbar}{2}, \quad \Delta E \Delta t \geq \frac{\hbar}{2}$$

- It can be derived using Cauchy-Schwarz inequality.