A short introduction to quantum mechanics II: why is a wave function a vector?

In our earlier article "Schrödinger equation," we introduced the concept of a wave function. It is a function of position, of which the magnitude squared is probability density function. On the other hand, in another article of ours "A short introduction to quantum mechanics I: observables and eigenvalues," we explained that a vector corresponding to each object in quantum mechanics is called "state vector" or "wave function." In other words, we claimed that wave functions were vectors.

Even though a claim that a function can also be a vector may seem unexpected, we will now explain why a wave function is a vector, and why multiplication by x and differentiation with respect to x are linear operators (or matrices).

A wave function is a vector because it satisfies the following eight conditions:

- 1) Vector addition must be associative. This condition is satisfied for wave functions $u(x)$, $v(x)$, and $w(x)$ because $u(x) + (v(x) + w(x)) = (u(x) + v(x)) + w(x)$.
- 2) Vector addition must be commutative. This condition is satisfied because $u(x) + v(x) = v(x) + u(x)$.
- 3) Vector addition must have an identity element. This condition is satisfied because $u(x) + 0 = u(x)$, where 0 denotes a constant function 0 for all x.
- 4) Vector addition must have inverse elements. This condition is satisfied because $u(x) + -u(x) = 0$, where $-u(x)$ is the additive inverse of $u(x)$.
- 5) Distributivity must hold for scalar multiplication over vector addition. This condition is satisfied because $a(u(x)+v(x)) = au(x)+av(x)$.
- 6) Distributivity must hold for scalar multiplication over field addition. This condition is satisfied because $(a + b)u(x) = au(x) + bu(x)$.
- 7) Scalar multiplication must be compatible with multiplication in the field of scalars. This condition is satisfied because $a(bu(x)) = (ab)u(x)$.

- 8) Scalar multiplication must have an identity element. This condition is satisfied because $1u(x) = u(x)$.

Now, I will explain why multiplication by x and differentiation with respect to x are linear operators (or matrices).

A linear operator (or matrix) L should satisfy the following conditions:

$$L(x+y) = L(x) + L(y)$$

$$L(ax) = aL(x)$$

Multiplication by x satisfies the above conditions because $x(u(x)+v(x)) = xu(x)+xv(x)$ and $x(au(x)) = axu(x)$. Differentiation with respect to x satisfies the above conditions because

$$\frac{\partial(u(x)+v(x))}{\partial x} = \frac{\partial u(x)}{\partial x} + \frac{\partial v(x)}{\partial x}$$

and

$$\frac{\partial(au(x))}{\partial x} = a\frac{\partial u(x)}{\partial x}$$

Therefore, multiplication by x and differentiation with respect to x are linear operators (or matrices).

Let us conclude this article with a comment. A vector is defined always in a vector space, and a vector space has always a dimension. Then, what is the dimension of the vector space in which a wave function considered in this article lives in?

Before answering this question, remember how a generic n-dimensional vector $\vec{v}$ can be expressed. If $\vec{e}_1, \vec{e}_2, \cdots, \vec{e}_n$ is the basis, we have

$$\vec{v} = \sum_{i=1}^{n} v_i \vec{e}_i \tag{1}$$

Here, we see that i is the label that denotes each basis $\vec{e}_i$ and v_is are the coefficients for $\vec{v}$.

Similarly, a wave function $u(x)$ can be regarded as the following vector:

$$\vec{u} = \int_{-\infty}^{\infty} dx\, u(x)|x\rangle \tag{2}$$

Here we see that x is the label that denotes each basis $|x\rangle$ and $u(x)$ is the coefficients for $\vec{u}$. Also, as x can be any value between $-\infty$ and ∞, we have integration instead of the sum as in (1). We see that the vector space a wave function lives in is infinite-dimensional, as there are infinitely many values for x which label the basis.

Summary

- A wave function is a vector because it satisfies the requirement of linearity.
- Multiplication by x and differentiation with respect to x are linear operators (i.e. matrices) because they satisfy the requirement of linearity.
- A wave function $u(x)$ can be regarded as the coefficients of a state vector $\vec{u}$ for the basis $|x\rangle$. The vector space is infinite-dimensional, as x can be any value between $-\infty$ and ∞.