A short introduction to quantum mechanics VIII: global gauge transformation

In our earlier article "A short introduction to quantum mechanics V: the expectation value of given observable," we have seen that the expectation value of a given observable "A" is given by the following formula.

$$< A >=< \psi|A|\psi > \tag{1}$$

Now, notice that the expectation value of A is invariant under the following transformation, called "global gauge transformation":

$$|\psi >\to e^{i\theta}|\psi > \tag{2}$$

where θ is a real number. (i.e. $e^{i\theta}$ is a pure phase) This is so, since the above transformation implies:

$$< \psi| \to e^{-i\theta} < \psi| \tag{3}$$

which, in turn, implies:

$$< \psi|A|\psi >\to< \psi|e^{-i\theta}Ae^{i\theta}|\psi >=< \psi|e^{-i\theta}e^{i\theta}A|\psi >=< \psi|A|\psi > \tag{4}$$

where $Ae^{i\theta}|\psi >= e^{i\theta}A|\psi >$ is satisfied because A is a linear operator and $e^{i\theta}$ is merely a number. As all the expectation values are invariant under global gauge transformation, we can conclude that a system described by the wave function $|\psi >$ is same as the one described by $e^{i\theta}|\psi >$. In other words, we say that we have a freedom to choose arbitrary "gauge" (i.e. we can choose an arbitrary θ) without altering physics.

It is also easy to see that Schrödinger equation is satisfied for the gauge transformed wave-function, if the original wave-function satisfies Schrödinger equation, as $(\frac{p^2}{2m} + V)|\psi >= E|\psi >$ implies $(\frac{p^2}{2m} + V)e^{i\theta}|\psi >= Ee^{i\theta}|\psi >$.

Finally, we conclude this article with a comment. Expectation values are not invariant under the following transformation, called "local gauge transformation".

$$|\psi >\to e^{i\theta(x)}|\psi > \tag{5}$$

where $\theta(x)$ is a function that depends on position x. Otherwise, one would have :

$$\frac{\partial\psi}{\partial x} \to e^{i\theta(x)}\frac{\partial\psi}{\partial x} \tag{6}$$

But (5) doesn't imply the above formula, since

$$\frac{\partial(e^{i\theta(x)}\psi)}{\partial x} = i\frac{\partial\theta(x)}{\partial x}e^{i\theta(x)}\psi + e^{i\theta(x)}\frac{\partial\psi}{\partial x} \neq e^{i\theta(x)}\frac{\partial\psi}{\partial x} \tag{7}$$

However, one can construct a theory that is invariant under local gauge transformation. We will see more of this in my article "What is a gauge theory?" Physicists usually call "local gauge transformation" simply "gauge transformation."

Problem 1. Can a global gauge transformation ever change the norm of a vector?

Problem 2. Why should θ must be a real number? Show that the norm changes if θ is not real.

Summary

- Physics is invariant under global gauge transformation $|\psi\rangle \to e^{i\theta}|\psi\rangle$.