Relativistic energy

In the last article, we have derived the following formula for relativistic momentum.

$$\vec{p} = m\vec{v} \tag{1}$$

or

$$\vec{p} = \frac{m_0\vec{v}}{\sqrt{1 - |\vec{v}|^2/c^2}} \tag{2}$$

where m_0 is the rest mass.

From the above formula we see that the momentum diverges (i.e. approaches infinity) as the velocity approaches the speed of light. Remembering that momentum is force integrated by time, we notice that you cannot make an object move faster than the speed of light once its mass is non-zero, since you would need to exert force on the object for an infinite amount of time.

Then, what would be the acceleration of an object with mass m if a force F is exerted? In the Newtonian case, it is F/m. In the relativistic case, it is modified. Nevertheless, the following formula is still obeyed:

$$\vec{F} = \frac{d\vec{p}}{dt} \tag{3}$$

This yields:

$$\begin{aligned} \vec{F} &= \frac{d}{dt}\left(\frac{m_0\vec{v}}{1 - \vec{v}\cdot\vec{v}/c^2}\right) \\ &= \frac{m_0}{\sqrt{1 - v^2/c^2}}\frac{d\vec{v}}{dt} + m_0\vec{v}\left(-\frac{1}{2}\right)(1 - v^2/c^2)^{-3/2}\left(-2\frac{\vec{v}}{c^2}\frac{d\vec{v}}{dt}\right) \\ &= \frac{m_0}{\sqrt{1 - v^2/c^2}}\frac{d\vec{v}}{dt} + \frac{m_0\vec{v}}{(\sqrt{1 - v^2/c^2})^3}\frac{d\vec{v}}{dt}\cdot\frac{\vec{v}}{c^2} \end{aligned} \tag{4}$$

Einstein didn't get this equation correct when he published his famous paper on special relativity in 1905. He calculated the force not from (2), but from a wrong method playing around with the Lorentz transformation. Planck obtained the equation (2) and correspondingly the correct equation for force in 1906.

Notice that in a simple one-dimensional situation (i.e. when the object moves along a straight line), the above formula becomes:

$$\begin{aligned} F &= \frac{d}{dt}\left(\frac{m_0 v}{1 - v^2/c^2}\right) \\ &= \frac{m_0}{\sqrt{1 - v^2/c^2}}\frac{dv}{dt} + m_0 v(-\frac{1}{2})(1 - v^2/c^2)^{-3/2}(-2\frac{v}{c^2}\frac{dv}{dt}) \\ &= \frac{m_0 v}{(\sqrt{1 - v^2/c^2})^3}\frac{dv}{dt} \end{aligned} \quad (5)$$

Given this, what would be the kinetic energy of an object moving with speed v? We can calculate as follows, exactly as in the Newtonian case:

$$\begin{aligned} \text{Kinetic Energy} &= \int F ds = \int \frac{m_0 v}{(\sqrt{1 - v^2/c^2})^3}\frac{dv}{dt} ds \\ &= \int \frac{m_0 v}{(\sqrt{1 - v^2/c^2})^3} dv \frac{ds}{dt} = \int \frac{m_0 v}{(\sqrt{1 - v^2/c^2})^3} dvv \\ &= \int_{v'=0}^{v'=v} \frac{m_0 v'^2}{(\sqrt{1 - v'^2/c^2})^3} dv' \\ &= \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2 \end{aligned} \quad (6)$$

Now, let's check that the above equation yields the Newtonian result $\frac{1}{2}m_0 v^2$ in the Newtonian limit, in which the speed v is much less than the speed of light c. We have:

$$\begin{aligned} \text{Kinetic Energy} &= \frac{m_0 c^2}{\sqrt{1 - v^2/c^2}} - m_0 c^2 = m_0 c^2 \left(\left(1 - \frac{v^2}{c^2}\right)^{-1/2} - 1\right) \\ &\approx m_0 c^2 \left(1 - \left(-\frac{1}{2}\frac{v^2}{c^2}\right) - 1\right) = m_0 c^2 \frac{v^2}{2c^2} = \frac{1}{2} m_0 v^2 \end{aligned} \quad (7)$$

where to get from the first line to the second, we used the binomial theorem. (Please read "The imagination in mathematics: Pascal's triangle, combination, and the Taylor series for square root" for a reminder on what the binomial theorem is.)

Given this, recall that the relativistic mass (or simply "mass" as many physicists call this) of a moving object m in terms of the rest mass m_0 and the speed v is given as follows:

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \quad (8)$$

In other words, the relativistic mass increases as it moves faster and faster, and it is equal to m_0 when it is at rest (i.e. $v = 0$). Then, we can write out the formula for the kinetic energy (6) as:

$$\text{Kinetic Energy} = mc^2 - m_0c^2 \tag{9}$$

Now comes the crucial point. Let's just say that the total energy of a particle with mass m is mc^2. This energy includes the kinetic energy ($= mc^2 - m_0c^2$) as well as what is called the "rest" energy ($= m_0c^2$). As it moves faster the mass will increase so its total energy will also increase, which in turn implies that the kinetic energy also increases as the rest energy is constant. Similarly, if it slows down, the mass will decrease, and so will the total energy and hence the kinetic energy as well. So, we can indeed write the total energy as follows:

$$E = mc^2 \tag{10}$$

which implies that mass and energy are the same, being proportional to each other. It also shows that a very small amount of mass can have very high energy since the proportionality factor c^2 is very large. However, it is a different matter whether the energy, especially the rest energy, can be easily converted to other forms of energy which we can use. This is actually possible in nuclear reactions or nuclear bombs. In these cases, the rest energy can be converted to energy that we can use. Again, as c^2 is very large, only something on the order of 1 gram of the rest energy was converted in the nuclear bombs dropped in Japan during World War II, which was enough to kill so many people.

In this article, we have introduced the concept of rest energy without a good justification other than the fact that introducing it made the sum of the kinetic energy and the rest energy possible to be expressed in a simpler manner. Nevertheless, in our later article "Mass-energy equivalence," we will introduce more justifications for the rest energy and the formula (10).

Now, there is another bonus for expressing the total energy as (10) and the momentum as (1). By explicit calculation using (8), we can show that the following is satisfied:

$$E^2 = (m_0c^2)^2 + (pc)^2 \tag{11}$$

Now, what would be the rest mass of a photon, the light particle? If the rest mass of a photon were given by a non-zero positive number, it would have infinite momentum and infinite energy as (6) and (2) show. As we know that it doesn't have infinite momentum and infinite energy, the only conclusion we can reach is that its rest mass is zero. This implies that m_0 is zero for the formula (11) in the case of a photon, and we conclude that the following relation between energy and momentum must be satisfied for a photon.

$$E = pc \tag{12}$$

Particle physicists usually call the rest mass simply “mass” and the relativistic mass “energy” by using the formula (10). For example, we often say that a photon is a “massless” particle, and quarks are “massive” particles.

Problem 1. In our earlier article “‘Group velocity and phase velocity” we have seen that the group velocity is given as follows:

$$v_g = \frac{d\omega}{dk} \tag{13}$$

Using $E = \hbar\omega$, $p = \hbar k$, we can write the above equation as:

$$v_g = \frac{dE}{dp} \tag{14}$$

Using (1), (10) and (11) show that the group velocity v_g is indeed the particle's speed v, also in the relativistic case.

Summary

- Relativistic mass is given by $m = \gamma m_0$.
- Relativistic energy is given by $E = mc^2$ where m is relativistic mass.
- Relativistic kinetic energy is given by $(\gamma - 1)m_0c^2$.
- $E^2 = (m_0c^2)^2 + (pc)^2$.
- For a photon, we can plug in $m_0 = 0$ for the above relation, and we get $E = pc$.