Relativistic momentum

In earlier articles, we saw that the absoluteness of time, length, and simultaneity was shattered by Einstein's theory of relativity. In this article, we will see how the formula for momentum is modified in the theory of relativity.

What is momentum? In Newtonian mechanics, if the mass is m, momentum is defined as follows:

$$\vec{p} = m\vec{v} \tag{1}$$

This formula holds still true in the theory of relativity, but with a caveat. It turns out that m depends on the speed of the object v. Such an $m(v)$ is called "relativistic mass" and the mass of the object when it is at rest (i.e. $v = 0$) is called "rest mass" and usually denoted as m_0. We will now find a relation between $m(v)$ by closely following Chapter 16 of "The Feynman Lectures on Physics, Vol. 1" available at `http://www.feynmanlectures.caltech.edu/I_16.html`.

First, assume that two objects A and B with each rest mass m_0 collide. Recalling what we have learned in "Elastic collision in 2-dimension" we can see the collision from the center of mass frame as in Fig.1. As explained in that article, in such a frame, the speeds of A before and after the collision are same, and the speeds of B before and after the collision are same. Furthermore, as the rest mass of A and B are same, it is not difficult to imagine that the speed of A before the collision and the speed of B before the collision are same as the total momentum in the center of mass frame is always zero; the momentum of B before the collision has the same magnitude as the momentum of A before collision but opposite direction. Of course, it goes without saying that the speed of A after the collision and the speed of B after the collision are same. All these speeds are denoted as v_0 in the figure.

Now, let's say that frame S moves with the same velocity as that of the x-component of object B. Then, frame S will observe that the velocities of B have only y-component, before and after the collision. See Fig.2. We call the speed of B as w and the velocity of A as v, and the angle the speed of A makes with x-axis as α. Let's also say that frame S' moves with the same velocity as that of the x-component of object A. Then, frame S' will observe that the velocities of A have only y-component, before and after the

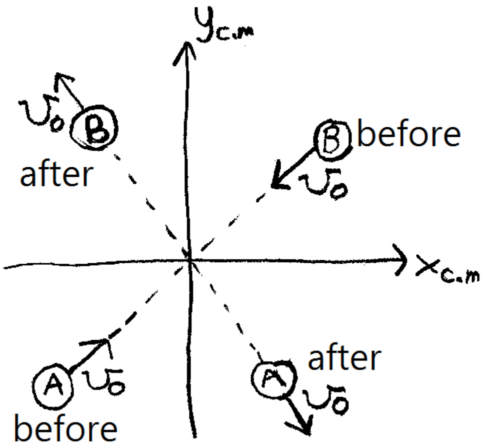

Figure 1: center of mass frame

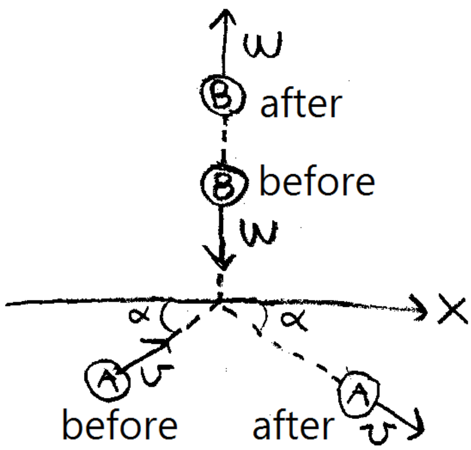

Figure 2: frame S

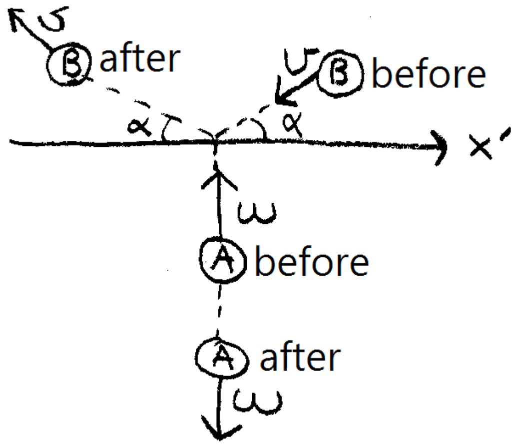

Figure 3: frame S'

collision. See Fig.3. Notice also that the speed of A is also w, which is same as the speed of B in the frame S and the speed of B is v, which is same as the speed of A in the frame S. The angle α is also same. They are all same because of the symmetry; the speeds of A and B were all the same in Fig.1.

What is the Lorentz transformation that relates the frame S and the frame S'? Let's call the x component of the velocity of object A by $u = v\cos\alpha$. Then, it is easy to see that the frame S' moves with speed u in positive x-direction with respect to the frame S. Therefore, we can write:

$$
\begin{aligned}
x' &= \gamma(x - ut) && (2)\\
y' &= y && (3)\\
z' &= z && (4)\\
t' &= \gamma(t - \frac{ux}{c^2}) && (5)\\
\gamma &= \frac{1}{\sqrt{1 - u^2/c^2}} && (6)
\end{aligned}
$$

which implies

$$
\begin{aligned}
v'_x &= \frac{v_x - u}{1 - \frac{uv_x}{c^2}} && (7)\\
v'_y &= \frac{v_y\sqrt{1 - u^2/c^2}}{1 - \frac{uv_x}{c^2}} && (8)\\
v'_z &= \frac{v_z\sqrt{1 - u^2/c^2}}{1 - \frac{uv_x}{c^2}} && (9)
\end{aligned}
$$

Given this, let's find out the y component of the velocity of B in the frame S'. (i.e. $v\sin\alpha$) We need to use (8). Using the fact that v_x of B in frame S is 0, and v_y of B in frame S is w, we have:

$$v\sin\alpha = w\sqrt{1 - u^2/c^2} \tag{10}$$

Now, in frame S', the momentum change of B upon collision is along y axis and given by

$$\Delta p = 2m(v)v\sin\alpha = 2m(v)w\sqrt{1 - u^2/c^2} \tag{11}$$

Similarly, the momentum change of A upon collision is given by

$$\Delta p = 2m(w)w \tag{12}$$

Equating them, as that is what the conservation of momentum means, we have:

$$m(v)\sqrt{1 - u^2/c^2} = m(w) \tag{13}$$

So what is v in terms of u and w? We have,

$$v^2 = (v\cos\alpha)^2 + (v\sin\alpha)^2 = u^2 + w^2(1 - u^2/c^2) \tag{14}$$

Now, let's think of the limit when $w = 0$. Plugging this value to (14), we get $v = u$. Then, upon using our definition of rest mass (i.e. $m_0 = m(0)$), (13) becomes

$$m(u) = \frac{m_0}{\sqrt{1 - u^2/c^2}} \tag{15}$$

This is the result we wanted to find. In conclusion, the momentum of a particle with rest mass m_0 and speed v is given by

$$p = m(v)v = \frac{m_0 v}{\sqrt{1 - v^2/c^2}} \tag{16}$$

In the next article, we will derive relativistic kinetic energy using this formula.

Summary

- The relativistic momentum is given by $p = \gamma m_0 v$ where m_0 is the rest mass.