$\sqrt{2}$ as irrational number

When we first learn numbers, we learn natural numbers (also called "positive integers") such as 1, 2, 3, $4 \cdots$. If we add a natural number to a natural number, we get a natural number. For example, $2 + 3 = 5$. 5 is a natural number. We say natural numbers are "closed" under addition. If we multiply a natural number by a natural number, we get a natural number. For example, $3 \times 4 = 12$. We say natural numbers are "closed" under multiplication. However, if we subtract a natural number from a natural number, sometimes we get a natural number, but sometimes we don't. For example, $4 - 3 = 1$. 1 is a natural number. On the other hand, there is no answer for $2 - 7$, if we confine the boundary of number to natural number. We say natural numbers are not "closed" under subtraction. However, if we introduce zero and negative integers such as $-1 - 2, -3, -4 \cdots$, we can assign an answer to $2 - 7$, which is -5. Now, we note that integers, which include positive integers, zero, and negative integers are closed under subtraction. What we mean is that we always get an integer if we subtract an integer from an integer. We have already seen an example, 2, 7 and -5 are all integers. Another example would be $-7 - (-8) = 1$. -7, -8, and 1 are all integers.

Problem 1. Are integers closed under addition? Are integers closed under multiplication?

If we divide an integer by another integer, sometimes we get an integer, but sometimes we don't. $12 \div 3 = 4$, but $18 \div 4$ doesn't have an answer, if we confine the boundary of number only to integers. Integers are not closed under division. But, if we are allowed to use fractions, we can find an answer to $18 \div 4$. It is $\frac{9}{2}$. Thus, we introduce what is called "rational number." By definition, a rational number is a number that can be expressed as $\frac{a}{b}$ where a and b are integers. $\frac{9}{2}$ is a rational number, because 9 and 2 are integers. It is also easy to see that all integers are rational numbers. For example, -4 is a rational number, because it can be expressed as $\frac{-4}{1}$. Rational numbers are closed under addition, subtraction, multiplication, and division. For example,

$$\frac{1}{2} + \frac{2}{3} = \frac{7}{6} \tag{1}$$

$$\frac{1}{5} - \frac{1}{4} = \frac{-1}{20} \tag{2}$$

$$\frac{-3}{4} \times \frac{8}{7} = \frac{-6}{7} \tag{3}$$

$$-4 \div \frac{2}{3} = -6 \tag{4}$$

It seems that everything is OK. If we have rational numbers, we can do all we want to do: addition, subtraction, multiplication and division. Perhaps from this reason, Pythagoras believed that all numbers were rational; we do not need any other numbers, because we could perhaps express every number in the form of $\frac{a}{b}$ where a and b are integers. Perhaps, these a and b might be big in practice. For example, $1.5436789333333333\cdots$ is $\frac{23155184}{15000000}$. But, in principle, this should not matter.

However, it is widely believed that Hippasus, Pythagoras's student proved that $\sqrt{2}$ is *not* a rational number. No matter how big a and b you choose, you can never express $\sqrt{2}$ as a/b where a and b are integers. For example, $\sqrt{2}$ is

$$\sqrt{2} = 1.414213562373095488\cdots \tag{5}$$

The rational number $\frac{707}{500} = 1.414$ is a good approximation of $\sqrt{2}$ but it is not equal to $\sqrt{2}$. $\frac{239}{169} \approx 1.414201$ is better, but it is not equal either. By choosing bigger a and b, we can get closer, as

$$\frac{665857}{470832} = 1.41421356237468991\cdots \tag{6}$$

but, nevertheless different.

Let me introduce a proof that $\sqrt{2}$ it not rational. We will use the method called "proof by contradiction." We will assume that $\sqrt{2}$ is rational, then show that this assumption leads to a contradiction, which means that our assumption was wrong. Before doing so, first, note that a rational number can be always expressed in the form of $\frac{c}{d}$ where c and d are integers that are relatively prime (i.e., there is no common factor but 1). Let me explain what I mean. Consider the rational number $\frac{12}{18}$. The greatest common factor of 12 and 18 is 6. Thus, we have

$$\frac{12}{18} = \frac{6 \times 2}{6 \times 3} = \frac{2}{3} \tag{7}$$

Now, this rational number is in the form we desired because the common factor of 2 and 3 is 1. In other words, we can always cancel the common factor that is not 1, to eventually make the common factor of the numerator and the denominator 1.

Now comes the proof. We will assume that $\sqrt{2}$ is rational. Then, we have $\sqrt{2} = \frac{c}{d}$ for some natural numbers c and d that are relatively prime. We have

$$(\sqrt{2})^2 = \left(\frac{c}{d}\right)^2 \tag{8}$$

$$2 = \frac{c^2}{d^2} \tag{9}$$

which means

$$c^2 = 2d^2 \tag{10}$$

. Thus, we see that c^2 is an even numbeer, because $2d^2$ is an even number. If c^2 is an even number, c is an even number. Otherwise, c^2 would have been odd. As c is an even number, we can express c as $c = 2f$, for some integer f. Now, let's plug this relation into (10). We get

$$c^2 = 4f^2 = 2d^2 \tag{11}$$

which means $d^2 = 2f^2$. Thus, we see that d^2 is an even number, which implies d is even.

In conclusion, both c and d are even. However, this contradicts the fact that c and d are relatively prime. If both c and d are even, they have the common factor 2. Thus, we conclude that our original assumption that $\sqrt{2}$ is rational is wrong. $\sqrt{2}$ is irrational.

Problem 2. Prove that $\sqrt{3}$ is irrational number.

Final comment. If you major in mathematics at a university, you learn "abstract algebra" in sophomore year. There, you learn things like the number $a + b\sqrt{3}$ where a and b are rational numbers are closed under addition, subtraction, multiplication and division. As an example for closure under addition, consider

$$\left(1 + \frac{1}{2}\sqrt{3}\right) + \left(\frac{2}{3} + \frac{1}{3}\sqrt{3}\right) = \frac{5}{3} + \frac{5}{6}\sqrt{3} \tag{12}$$

Note here that 1, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{1}{3}$, $\frac{5}{3}$ and $\frac{5}{6}$ are all rational numbers. It is very easy to see that the number we considered (i.e., $a + b\sqrt{3}$) is closed under addition and subtraction. If you want to challenge yourself, convince yourself that they are also closed under multiplication and division. Multiplication won't be that difficult, but division will be a bit hard, if you are not familiar with manipulating square root.