Rotation and the Lorentz transformation, orthogonal and unitary matrices

In our earlier article, "Lorentz transformation and Rotation, a comparison," I explained that proper time and proper distance are invariant under Lorentz transformation and the length is invariant under rotation. In this article, we go one step further and present conditions the Lorentz transformation and Rotation matrix should satisfy.

To this end, let's express proper time in matrix form. If we let:

$$\vec{v} = \begin{pmatrix} c\Delta t \\ \Delta x \\ \Delta y \\ \Delta z \end{pmatrix}, \quad \eta = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

Then, the proper time is given by

$$(\Delta\tau)^2 = v^T \eta v \tag{2}$$

Now, if we define Λ, the Lorentz transformation matrix, as follows,

$$\Lambda \equiv \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3}$$

then we can express the Lorentz transformation in matrix form as follows:

$$\vec{v}' \equiv \begin{pmatrix} c\Delta t' \\ \Delta x' \\ \Delta y' \\ \Delta z' \end{pmatrix} = \Lambda v \tag{4}$$

Our earlier article comparing Lorentz transformation and rotation found that:

$$(v')^T \eta v' = v^T \eta v \tag{5}$$

Plugging (4) into to the above equation yields:

$$\Lambda^T \eta \Lambda = \eta \tag{6}$$

Similarly, if we let the rotation matrix be O and take similar steps to those we have just taken with Lorentz transformation matrix, we get:

$$O^T I O = I \tag{7}$$

where I is the identity matrix. This in turn, implies:

$$O^T O = I \tag{8}$$

A matrix that satisfies the above condition is called an "orthogonal matrix." (**Problem 1.** Prove that the determinant of an orthogonal matrix is always either 1 or -1.)

We denote an orthogonal matrix of size $n \times n$ by $O(n)$. For example, the matrix

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{9}$$

from our article "Rotation in Cartesian coordinate" is an example of an $O(2)$ matrix; it is a 2×2 matrix which expresses a rotation in two dimensions. While all rotation matrices are orthogonal matrices, not all orthogonal matrices correspond to rotation matrices. For example, the following matrix is not a rotation matrix:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{10}$$

It sends (x, y) to $(x, -y)$. Therefore, it is a matrix that represents reflection about the x-axis. Notice also that the determinant of this matrix is -1, not 1 as in (9). It can be shown that all rotation matrices have determinant 1. Think it in this way: an identity matrix is a rotation matrix with determinant 1; it rotates a vector by zero degree. Now notice that every rotation matrix rotates a vector by a certain angle. For example let's say O_θ is a matrix that rotates a vector by θ degrees about a particular axis. We know that the determinant of O_θ has to be either 1 or -1 because it is an orthogonal matrix and that it must be 1 for $\theta = 0$. As θ gradually increases from 0, the determinant cannot suddenly jump from 1 to -1, as all the numbers in the rotation matrix change gradually without sudden jumping (we say that the matrix O_θ is "connected" to the identity matrix). Therefore, we conclude that the determinant of a rotation matrix is 1.

An orthogonal matrix with determinant 1 is called a "special orthogonal matrix." Such an $n \times n$ matrix is denoted as $SO(n)$. Similarly, we call the matrix that satisfies (6) with determinant 1, $SO(1,3)$. Here $(1,3)$ denotes the fact that η has one 1 and three -1s in the diagonal part of η. (i.e. 1 positive eigenvalue and 3 negative eigenvalues) (**Problem 2.** Prove that the determinant of the matrix that satisfies (6) is always either 1 or -1. Also, check that the determinant of (3) is 1.)

Notice that an orthogonal matrix necessarily preserves the dot product. For example, if $v_1' = Ov_1$, $v_2' = Ov_2$, we have:

$$v_1' \cdot v_2' = (v_1')^T v_2' = (v_1)^T O^T O v_2 = v_1^T v_2 = v_1 \cdot v_2 \tag{11}$$

Similarly, if we define the dot product of two "4-vectors" $\vec{A}$, $\vec{B}$ in special relativity as follows,

$$\begin{aligned}\vec{A} \cdot \vec{B} &= (A_t\hat{t} + A_x\hat{x} + A_y\hat{y} + A_z\hat{z}) \cdot (B_t\hat{t} + B_x\hat{x} + B_y\hat{y} + B_z\hat{z}) \\ &= A_tB_t - A_xB_x - A_yB_y - A_zB_z \end{aligned} \tag{12}$$

and set $v_1' = \Lambda v_1$, $v_2' = \Lambda v_2$, we easily see that Lorentz transformation also preserves the dot product as:

$$v_1' \cdot v_2' = (v_1')^T \eta v_2' = (v_1)^T \Lambda^T \eta \Lambda v_2 = v_1 \eta v_2 = v_1 \cdot v_2 \tag{13}$$

We will have a chance to talk more about 4-vector in another article. As we have seen in our seventh article on quantum mechanics, unitary matrices are often more useful than orthogonal matrices in quantum mechanics.

Problem 3. Show that (3) can be re-expressed as

$$\Lambda \equiv \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{14}$$

where $\tanh\phi = v/c$ (such a ϕ is called the "rapidity" of the transformation). Hint[1]. Written this way, the transformation matrix looks quite similar to (9). In other words, hyperbolic sine and cosine play a similar role in the Lorentz transformation to sine and cosine in a rotation. Also, check that the determinant of (14) is 1.

Problem 4. If you rotate a vector by θ_1 around a certain axis and rotate it again around the same axis by θ_2, the result is a rotation by $\theta_1 + \theta_2$. This can be seen as follows:

$$\begin{pmatrix} \cos\theta_2 & -\sin\theta_2 \\ \sin\theta_2 & \cos\theta_2 \end{pmatrix} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 \\ \sin\theta_1 & \cos\theta_1 \end{pmatrix} = \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) \end{pmatrix}$$

Roughly speaking, you "boost" a vector in a Lorentz transformation instead of rotating it. Check for yourself that boosting a vector with a rapidity ϕ_1 along the x-direction and boosting it again with a rapidity ϕ_2 along the x-direction is the same thing as boosting with a rapidity $\phi_1 + \phi_2$ in a single go.

Also, from $\tanh(\phi_1 + \phi_2) = (\tanh\phi_1 + \tanh\phi_2)/(1 + \tanh\phi_1 \tanh\phi_2)$, derive the relativistic addition rule for velocities.

[1]Try to express $\cosh\phi$ and $\sinh\phi$ in terms of $\tanh\phi$, using $\cosh^2\phi - \sinh^2\phi = 1$ and $\tanh\phi = \sinh\phi/\cosh\phi$