Row reduction and echelon form

As promised in our earlier article "System of linear equations, part II: three or more unknowns," we will mathematically prove that we need exactly n linearly independent equations if we have n unknowns.

To this end, we will first express a system of linear equations using matrices. For example,

$$
\begin{aligned}
2x + y - z &= 3 && (1)\\
x + y + 2z &= -1 && (2)\\
2x - 2y + z &= 1 && (3)
\end{aligned}
$$

can be expressed as

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & 2 \\ 2 & -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} \tag{4}$$

or, in a simple short-hand notation,

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 1 & 1 & 2 & -1 \\ 2 & -2 & 1 & 1 \end{array}\right] \tag{5}$$

Now, we can obtain the unknowns by "row reducing" the matrix, using "row operations":

(1) Multiplying a row by a non-zero number.

(2) Adding a multiple of row to another row.

(3) Exchanging two rows

It goes without saying that row operations do not change the solution, as this is the usual procedure to find the solution in a system of linear equations as in the earlier article. Only the notation is different.

Let's solve (5) using row operations. We will first eliminate x, then y. If you multiply the first row by $-1/2$ and add it to the second row, you get $\left[\begin{array}{ccc|c} 0 & 1/2 & 5/2 & -5/2 \end{array}\right]$. If you multiply the first row by -1 and add it to the third row, you get $\left[\begin{array}{ccc|c} 0 & -3 & 2 & -2 \end{array}\right]$. Therefore, (5) becomes:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & 1/2 & 5/2 & -5/2 \\ 0 & -3 & 2 & -2 \end{array}\right] \tag{6}$$

So, we successfully eliminated x. That the first components of the second row and the third raw are zero tells that. Now, we can multiply the second row by 6 and add it to the third row to get:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & 1/2 & 5/2 & -5/2 \\ 0 & 0 & 17 & -17 \end{array}\right] \tag{7}$$

So, we successfully eliminated y. Now, if we divide the third row by 17, we get

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & 1/2 & 5/2 & -5/2 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{8}$$

Therefore, $z = -1$. If we multiply the third row by $-5/2$ and add it to the second row, then, multiply by 2 (which is equivalent to plugging $z = -1$ into the second row), we get:

$$\left[\begin{array}{ccc|c} 2 & 1 & -1 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{9}$$

By taking a similar procedure (which is equivalent to plugging $y = 0$ and $z = 1$ in the first row), we finally obtain:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right] \tag{10}$$

In other words,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{11}$$

We see that the first matrix in the left-hand side is the identity matrix. Therefore, we have:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \tag{12}$$

This is the solution.

Let's see what we have done. We row reduced the matrix (5) into (10) using row operations. In particular, we got an identity matrix on the left side of the bar.

However, for other examples of systems of linear equations, it can happen that we do not get an identity matrix there, even though we tried our best. In such a case, there can be no solution or infinite solutions depending on the situation. In any case, we say that a matrix is in echelon form, if the matrix is the one that we obtained after doing our best. Of course, a matrix that has identity matrix in it such as (10) is also in echelon form.

Let's define echelon form by following *Vector calculus, Linear Algebra, and Differential Forms: A Unified Approach* by John H. Hubbard and Barbara Burke Hubbard. A matrix is in echelon form if

(1)In every row, the first nonzero entry is 1, called a pivotal 1.

(2)The pivotal 1 of a lower row is always to the right of the pivotal 1 of a higher row.

(3)In every column that contains a pivotal 1, all other entries are 0.

(4)Any rows consisting entirely of 0's are at the bottom.

Here are some examples of matrices in echelon form. The pivotal 1s are in bold type:

$$\left[\begin{array}{cccc|c} \mathbf{1} & 0 & 1 & 0 & 1 \\ 0 & \mathbf{1} & 1 & 0 & 2 \\ 0 & 0 & 0 & \mathbf{1} & -1 \end{array}\right], \quad \left[\begin{array}{ccc|c} \mathbf{1} & 0 & 0 & 1 \\ 0 & \mathbf{1} & 0 & -2 \\ 0 & 0 & \mathbf{1} & -1 \\ 0 & 0 & 0 & \mathbf{1} \end{array}\right], \quad \left[\begin{array}{cccc|c} \mathbf{1} & 0 & 0 & -1 & -1 \\ 0 & \mathbf{1} & 0 & 3 & 2 \\ 0 & 0 & \mathbf{1} & 1 & 4 \end{array}\right] \tag{13}$$

Here are some examples of matrices that are not in echelon form.

$$\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 & 1 \end{array}\right], \quad \left[\begin{array}{ccc|c} 2 & 0 & 0 & 4 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{array}\right], \quad \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1 & -1 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 3 & 4 \end{array}\right] \tag{14}$$

In other words, they can be further reduced to echelon form by row operations. For example, in the third row of the first matrix, we have the pivotal 1 in the third entry. But the other two numbers (i.e. 1 and 2) in the third column are not zero. Therefore (3) in our definition is violated. We can bring it to echelon form by subtracting multiples of the third row from the first row and the second row.

Problem 1. Bring the three matrices in (14) to echelon form.

There is a theorem that says that we always obtain a unique matrix $\tilde{A}$ in echelon form from a matrix A by row operations. The proof is by construction. If we take the same step that led to (10), we will get a certain and unique matrix in echelon form.

Now, as promised in the earlier article, let's see what happens if there are less equations than unknowns. If we make the system of linear equations into echelon form, we can never have only identity matrix on the left-side of the bar as the number of row is smaller than the number of column. We must have an extra column or extra columns as in the fourth column (i.e. [-1,3,1]) in the third matrix of (13) or as in the third column (i.e. [1,1,0]) in the first matrix (13). This extra column make the unknowns under-determined. For example, if we denote the variables for these two matrices by x, y, z, w, the first row of the first matrix implies $x + z = 1$. So, we can choose any xs and zs that satisfy it as solutions. Similarly, the third row of the third matrix implies $z + w = 4$. We can choose any zs and ws that satisfy this as solutions.

Let's see what happens when there are more equations than unknowns. If we make it into echelon form as before, we can never have only identity matrix on the left-side of the bar as the number of row is bigger than the number of column. We must have an extra row or extra rows. A good example is the second matrix of (13). The extra row is $\left[\begin{array}{ccc|c} 0 & 0 & 0 & 1 \end{array}\right]$. If we denote the variables by x, y, z, it means

$$0x + 0y + 0z = 1 \tag{15}$$

which is impossible to satisfy for any x, y and z. So, there is no solution.

In conclusion, as we have loosely explained in the article on system of linear equations, we need as many linearly independent equations as unknowns to have a unique solution; even though the number of equations and the number of unknowns are same, there can't be a unique solution if the echelon form is the following form.

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{array}\right], \quad \left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right] \tag{16}$$

In the first case in the above equation, there are no solutions. In the second case, there are infinitely many solutions.

I visited Germany in summer 2000, and sat on high school classes. I found out that they taught matrix method to solve system of linear equations just like in this article. One of the other Korean friends told me that she couldn't see the reason why they solve system of linear equations using such a method. She may be right at high school level, but as you saw in this article, the method gives a powerful tool to systematically understand system of linear equations.

Summary

- A system of linear equations can be expressed using a matrix multiplication.
- We can solve a system of linear equations expressed such by bringing it to "echelon form" using row reductions.
- If the echelon form obtained such has the identity matrix, on the left part, the system of linear equations has a unique solution. Otherwise, infinite solutions or no solution at all.