The speed of sound

As promised in the last article, we will derive the speed of sound in this article. Remember that we mentioned there that the propagation of sound was just like the one of the spring wave. Therefore, to obtain the speed of sound, we will obtain the "spring constant" of the air. Then, we will plug this value to the formula for the speed of the spring wave, which you have obtained at the end of our last article.

To this end, we will regard the sound wave as if it is moving along cylinder or, in other words, inside piston. The spirit is similar to the one in our earlier article "Specific heats of gases." See Fig.1 in that article. There, we introduced the area of cover "A" and tried to calculate the work done by the gas. Nevertheless, in the final expression A disappeared and the result was given in terms of the pressure p and volume V. We will see that A, the area of cross section of the piston, eventually disappears in our case as well.

Given this, notice that a spring that is additionally stretched by the length dL feels an additional force of $dF = -KdL$. Similarly, if air is stretched by additional volume dV, we have,

$$dF = AdP = A\frac{dP}{dV}dV \tag{1}$$

Using $dV = AdL$, we get

$$dF = A^2\frac{dP}{dV}dL \tag{2}$$

Therefore, we can regard the spring constant of the air as

$$K = -A^2\frac{dP}{dV} \tag{3}$$

In the last article, if you solved the problem correctly, you obtained

$$v = \sqrt{\frac{K}{M}}L \tag{4}$$

Here, M is the mass of each object. In terms of the mass per length "λ" it is given by $M = \lambda L$. In our case, we need to use $M = \rho V$ where ρ is the mass density of air. Plugging this relation along with (3) to the above formula, we get,

$$v = AL\sqrt{-\frac{1}{\rho V}\frac{dP}{dV}} = \sqrt{-\frac{V}{\rho}\frac{dP}{dV}} \tag{5}$$

This was first derived by Newton. Newton, then, proceeded to use Boyle's law. Boyle's law implies,

$$0 = d(PV) = PdV + VdP \tag{6}$$

which, in turn, implies,

$$\frac{dP}{dV} = -\frac{P}{V} \tag{7}$$

Therefore, Newton obtained

$$v = \sqrt{\frac{P}{\rho}} \tag{8}$$

However, the value so obtained (979 feet per second) was about 15 percents lower than the value then recently measured. To remedy the situation, in the early 18th century, Newton proposed that the size of the particles of air was about one tenth their separation, which would increase the speed of sound by about 10 percent and the vapor in the air would increase another 5 percent.[1]

However, in the 18th century, physicists dismissed the Newton's explanation for the 15 percent difference between the measurement and the theory, which was then called the speed-of-sound problem. Many, including Euler and Lagrange, tried to solve the problem, but failed.

Laplace came up with the correct solution in the early 19th century. Newton's mistake was using Boyle's law, which is only satisfied for isothermal (i.e. constant temperature) process. During the compression and expansion of air due to sound wave passing, there is no enough time for the heat to transfer, which makes the process adiabatic. In such a case, we have (**Problem 1.** Check this!)

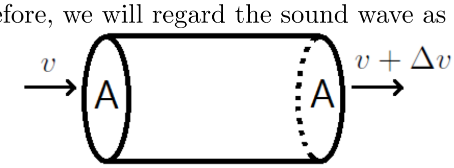
$$\frac{dP}{dV} = -\frac{\gamma P}{V} \tag{9}$$

Therefore, we have

$$v = \sqrt{\frac{\gamma P}{\rho}} \tag{10}$$

As the ratio P/ρ depends on temperature, the speed of sound depends on temperature. At 20°C, it is about 340 m/s.

We just derived the speed of sound by using the model of travelling spring wave. However, there are other ways to derive it without using that model. We will now introduce one. As before, we will regard the sound wave as if it is moving along a cylinder.



See the figure. The flux coming from the left-side must be equal to the flux going out to the right-side. (Remember our earlier discussion in the article "Continuity equation.") Therefore, we must have

$$\rho v A = (\rho + \Delta\rho)(v + \Delta v)A \tag{11}$$

where ρ and v are the density of the air and the speed of the air at the left-side of the cylinder and $\rho + \Delta\rho$ and $v + \Delta v$ are the ones at the right-hand side of the cylinder. From the above

[1]Finn, B. S.. (1964). Laplace and the Speed of Sound. Isis, 55(1), 719. Retrieved from http://www.jstor.org/stable/227751

equation, we get

$$\rho dv = -v d\rho \tag{12}$$

Of course, the above relation can be equally obtained from $d(\rho v)A = 0$ which follows from (11).

Now, let's consider the pressures on the cylinder and Newton's second law. On the left-side of the cylinder we have the force PA on the air in the cylinder while on the right-side we have the force $(P + \Delta P)A$. Therefore, the net force is

$$\Delta F = PA - (P + \Delta P)A = -\Delta PA = -\frac{\Delta P}{\Delta x}\Delta x A \tag{13}$$

The net force acts on the air inside the cylinder whose mass is $\rho A \Delta x$ and the acceleration $\Delta v/\Delta t$. Therefore, we get

$$\Delta F = -\frac{\Delta P}{\Delta x}\Delta x A = \rho A \Delta x \frac{\Delta v}{\Delta t} = \rho A v \Delta v \tag{14}$$

which simplifies to

$$-\Delta P = \rho v \Delta v \tag{15}$$

Plugging (12) we get

$$v = \sqrt{\frac{dP}{d\rho}} \tag{16}$$

Now, this leads to (10). (**Problem 2.** Check this!)

There is still another way to derive the speed of sound which I learned from a cosmology textbook. There, they derived the partial differential equation for the density wave, which is of the form

$$\frac{\partial^2 \rho}{\partial t^2} = v^2 \left(\frac{\partial^2 \rho}{\partial x^2} + \frac{\partial^2 \rho}{\partial y^2} + \frac{\partial^2 \rho}{\partial z^2} \right) \tag{17}$$

If you remember our earlier article "Separation of variables method in partial differential equations," you will see that this is exactly the wave equation. Anyhow, in this article, we will not introduce the derivation of the above equation that I learned from the cosmology textbook. Let us sketch the basic idea though. The idea is that you use the more general continuity equation

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \vec{v}) = 0 \tag{18}$$

instead of the continuity equation (11), and similarly for Newton's second law. This approach is more general and relies on less assumptions.

Summary

- Considering adiabatic expansion and contraction, one can calculate the speed of sound.