Surface area of n-sphere and volume of n-ball

In our earlier article "Polar coordinate, the area of a circle and Gaussian integral," we calculated the value for the Gaussian integral. There, we had to use the fact that the whole angle is 2π. Let's look at this slightly differently. If we didn't know that the whole angle was 2π, but knew the value for the Gaussian integral, we would be able to derive that the whole angle must be 2π. If you also remember that the length of a circle (i.e. 1-sphere) with radius r is $2\pi r$, precisely because the whole angle is 2π, we can say that we can deduce the length of a circle from the Gaussian integral. Stepping further, we can actually use the Gaussian integral to get the area of n-sphere for any n. Then, just as we could get the area of a 2-ball (i.e. disc) by integrating the length of 1-sphere (i.e. circle), we can get the volume of $(n+1)$-ball by integrating the area of n-sphere.

We will explicitly do this for 2-sphere, and leave the generalization to the readers.

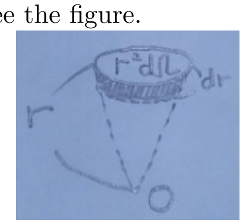
First, we have

$$\int e^{-(x^2+y^2+z^2)} dxdydz = \int e^{-x^2} dx \int e^{-y^2} dy \int e^{-z^2} dz = \pi^{3/2} \tag{1}$$

If we set $r^2 = x^2 + y^2 + z^2$, and use the fact that the volume element is given by

$$r^2 \, d\Omega \, dr = dx \, dy \, dz \tag{2}$$

See the figure.



Then, (1) is equal to

$$\begin{aligned} \int e^{-r^2} \int r^2 d\Omega dr &= \int d\Omega \int e^{-r^2} r^2 \, dr \\ &= \int d\Omega \, \Gamma\left(\frac{3}{2}\right) \times \frac{1}{2} \end{aligned} \tag{3}$$

Problem 1. Check this! (Hint: Use the integration by substitution and the definition of the gamma function introduced in "Gamma function." `http://youngsubyoon.com/Gamma.pdf`)

In conclusion, we get

$$\int d\Omega = \frac{2\pi^{3/2}}{\Gamma\left(\frac{3}{2}\right)} \tag{4}$$

Problem 2. Check that the above is indeed equal to 4π. (Hint:[1])

Therefore, the surface area of 2-sphere is given by

$$\int r^2 d\Omega = 4\pi r^2 \tag{5}$$

and the volume of 3-ball is given by

$$\int r^2 d\Omega dr = \int 4\pi r^2 dr = \frac{4}{3}\pi r^3 \tag{6}$$

Problem 3. Show that the surface area of n-sphere is given by

$$S_n(r) = \frac{2\pi^{(n+1)/2}}{\Gamma\left(\frac{n+1}{2}\right)} r^n \tag{7}$$

and the volume of n-ball is given by

$$V_n(r) = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)} r^n \tag{8}$$

In our later article on quantum field theory, we will have an occasion to use the formula (7) for a non-integer n. Of course, we didn't prove that (7) is valid for a non-integer n, but we can assume so from "analytic continuation." (See our earlier article "$1+2+3+4+\cdots = -1/12$" for another example of analytic continuation.)

Summary

- Surface area of n-sphere and volume of n-ball can be expressed by using Gamma functions.

[1] Use $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$, and $\Gamma(x+1) = x\Gamma(x)$.