Tangent lines to parabolas and circles

In our earlier article "The Cartesian coordinate system and graph," we have considered the intersection of two lines. In this article, we will first consider the intersection of a parabola and a line, and then the intersection of a circle and a line.

For the parabola consider $y = x^2$ and for the line consider $y = x + 2$. See Fig.1. Let's find their intersections. We get

$$\begin{aligned} x^2 &= x + 2 && (1) \\ x^2 - x - 2 &= 0 \\ x &= 2, -1 && (2) \end{aligned}$$

So, the x-coordinates of the intersections are $2, -1$. To get the y-coordinates, plug in these numbers to $y = x^2 = x + 2$. We get

$$2^2 = 2 + 2 = 4, \qquad (-1)^2 = -1 + 2 = 1 \tag{3}$$

So the intersections are at $(2, 4)$ and $(-1, 1)$. Notice that we have found two intersections, as there are two solutions to (1), which is a quadratic equation. So what was the original reason that (1) was a quadratic equation? It is because $y = x^2$, a parabola, is a quadratic expression, and $y = x + 2$ is a linear expression. Thus, we see that there can be two intersections of a parabola and a line. Of course, this is not always the case. As there can be one solution or no solutions at all for a quadratic equation, there can be one intersection or no intersections of a parabola and a line.

Problem 1. Show that there is no intersection of $y = x^2 + 2$ and $y = -2x$.

Now, let's consider the case that there is one intersection. For the parabola consider $y = \frac{1}{2}x^2$, and for the line consider $y = 2x - 2$. See Fig. 2. We can find the intersection as follows

$$\begin{aligned} \frac{1}{2}x^2 &= 2x - 2 && (4) \\ x^2 - 4x + 4 &= 0 && (5) \\ x &= 2 && (6) \end{aligned}$$

For the y-coordinate, we have

$$\frac{1}{2}2^2 = 2 \times 2 - 2 = 2 \tag{7}$$

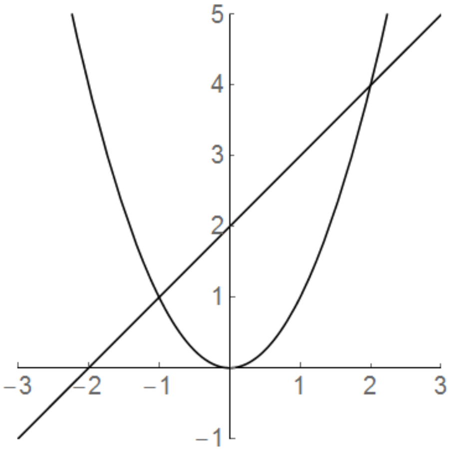

Figure 1: $y = x^2, x+2$

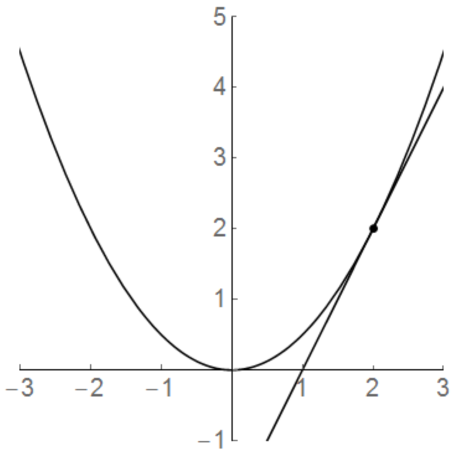

Figure 2: $y = \frac{1}{2}x^2, 2x-2$

So, $(2,2)$ is the intersection. Now, notice what the graphs look like. The line "touches" the parabola at the intersection $(2,2)$, and the slope at the intersection is equal to the slope of the line. We say the line $y = 2x-2$ is the tangent line to the parabola $y = \frac{1}{2}x^2$ at $(2,2)$. Now, remember why we came to such a conclusion. It's because the discriminant of (5) is 0. Actually, for a given parabola, we can apply the lesson learned here to find the tangent line that passes a certain point we choose. For example, for the parabola $y = x^2+2x$, let's find the tangent line that passes $(-1,-2)$. See Fig.3. First, notice that a line can be expressed as $y = ax+b$. If the line passes $(-1,-2)$, it can be expressed as a form $y = a(x+1)-2$. (**Problem 2.** Check this![1]) Indeed if you plug in $x = -1$, you get $y = a \cdot 0 - 2 = 0 - 2 = -2$ as expected. Then, to find the intersection of this line and $y = x^2+2x$, we can equate them as follows

$$\begin{aligned} x^2 + 2x &= a(x+1) - 2 \qquad (8) \\ x^2 + (2-a)x + 2 - a &= 0 \qquad (9) \end{aligned}$$

Now, let's impose the condition that the discriminant of (9) is 0, because we want exactly one intersection. We have,

$$(2-a)^2 - 4(2-a) = 0 \qquad (10)$$

You can solve this in two ways. In the first way, you expand everything to

[1] This is exactly the same problem as Problem 3 of "The Cartesian coordinate system and graph."

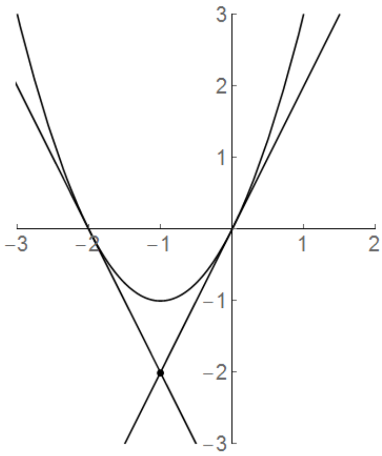

Figure 3: $y = x^2 + 2x, (-1, -2)$

get

$$\begin{aligned} 4 - 4a + a^2 - 8 + 4a &= 0 \quad (11) \\ a^2 - 4 &= 0 \\ a &= 2, -2 \quad (12) \\ & \quad (13) \end{aligned}$$

This way always works. In the second way, you can take advantage of the fact that $(2 - a)$ is repeated and factor it out. You have,

$$\begin{aligned} (2-a)(2-a) - 4(2-a) &= 0 \quad (14) \\ (2-a)(2-a-4) &= 0 \\ (2-a)(-2-a) &= 0 \quad (15) \end{aligned}$$

As either $2 - a = 0$ or $-2 - a = 0$, we conclude again $a = 2, -2$. Therefore, there are two tangent lines. The first one is given by

$$y = 2(x + 1) - 2 = 2x + 2 - 2 = 2x \quad (16)$$

The second one is given by

$$y = -2(x + 1) - 2 = -2x - 4 \quad (17)$$

See Fig.3 again.

Problem 3. Show that for a parabola $y = x^2 - 2x$ there is exactly one tangent line that passes (0,0). (Note: If you draw the parabola this will seem obvious as this point is on the parabola.)

Problem 4. Show that for a parabola $y = x^2$ there is no tangent line that passes (0,1). (Note: if you draw the parabola this will seem obvious as this point is inside the parabola.)

We can actually use the same method to find the intersection of a circle and a line, as well as a tangent line to the circle.

Problem 5. Find the intersection of $x^2 + y^2 = 1$ and $y = 1 - x$, draw the graphs of these two equations (i.e. a circle and a line,) and mark the intersections.

Problem 6. Show that $x^2 + y^2 = 2$ and $y = 2 - x$ intersect at one point. This shows that $y = 2 - x$ is the tangent line to $x^2 + y^2 = 2$. Draw the graphs of these two equations (i.e. a circle and a line) and mark the intersection.

Final comment. In this article, we used the discriminant to find the tangent line of parabolas and circles. However, there is a much easier way to find the tangent line. Moreover, you can use this method to find the tangent line of any mathematical function not just parabolas and circles. For example, using this method, you can find a tangent line to a complicated function such as $y = 1 + x^4 + \frac{1}{1-x}$. We will teach you this method in our later articles on calculus.

Summary

- A straight line intersects a parabola or a circle at two points, if the discriminant of the quadratic equation obtained to find the intersection is positive, at one point, if the discriminant is zero, and zero point, if the discriminant is negative.

- When the straight line "touches" the parabola (or the circle) at a certain point A, the slope at the intersection is equal to the slope of the line, and the straight line is called the tangent line of the parabola (or the circle) at point A.