Transition from classical mechanics to quantum mechanics

Remember that a commutator between two matrices (i.e. linear operators) A and B is defined as $[A,B] \equiv AB - BA$.

Notice that this commutator has a very similar structure to the Poisson bracket. Namely,

$$[A,B] = -[B,A] \tag{1}$$

and

$$[A,B+C] = [A,B] + [A,C] \tag{2}$$

$$[A,BC] = B[A,C] + [A,B]C \tag{3}$$

which can be checked very easily.

Furthermore, from our third article on quantum mechanics, we know that

$$[q_i, p_j] = i\hbar\delta_{ij} \tag{4}$$

whereas we had the following in our article on Poisson bracket.

$$\{q_i, p_j\} = \delta_{ij} \tag{5}$$

Given these comparisons, one can suspect that the following relation holds between

$$[A,B] = i\hbar\{A,B\} \tag{6}$$

This is true. Transition from classical mechanics to quantum mechanics, which is also called "quantization" takes place by replacing the Poisson bracket by the commutator with extra $i\hbar$ as the above equation:

For example, this gives the following equation of motion in quantum mechanics:

$$\frac{dA}{dt} = \{A,H\} = \frac{[A,H]}{i\hbar} \tag{7}$$

where A is an operator, not a number. So, physicists take the following procedure to quantize a theory.

First, we write the Lagrangian in terms of q and $\dot{q}$s. Then, we obtain conjugate momentums ps of the conjugate variable qs, by the following method, as mentioned in our earlier article:

$$\dot{p}_i = \frac{\partial L}{\partial q_i} \tag{8}$$

Then we say that the commutator between the conjugate momentum and the conjugate variable is $i\hbar$, as follows:

$$[q_i, p_i] = i\hbar \tag{9}$$

Physicists use this method to quantize all known theories such as string theory or loop quantum gravity. (There are other equivalent methods to quantize theories such as "Path integral quantization" which uses the Lagrangian and the action. We will talk more about it in our later articles. We will also see that the various quantization methods lead to the same results.)

Problem 1. In our third article on quantum mechanics, we have seen

$$[f(X), P_x] = i\hbar \frac{\partial f(x)}{\partial x} \tag{10}$$

By calculating

$$\{f(x), p_x\} = \frac{\partial f(x)}{\partial x} \tag{11}$$

check that (6) is indeed satisfied.

Summary

- Transition from classical mechanics to quantum mechanics, called "quantization," takes place by replacing the Poisson bracket by the commutator with extra $i\hbar$ factor.

- In more general cases such as string theory or loop quantum gravity, physicists quantize a system by imposing $[q_i, p_j] = i\delta_{ij}$ where p_j is the conjugate momentum of q_j.