The transpose and Hermitian conjugate

Transpose is an important operation for matrices. Using components, the transpose of a matrix A is defined as follows:

$$(A^T)_{ij} = A_{ji} \tag{1}$$

In other words, the transpose occurs when the column and row of a matrix are exchanged. Here is an example:

If A is defined as follows:

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 3 \\ 3 & 5 \end{pmatrix} \tag{2}$$

Then, we have:

$$A^T = \begin{pmatrix} 1 & 2 & 3 \\ -1 & 3 & 5 \end{pmatrix} \tag{3}$$

A useful property of the transpose operation is that:

$$(AB)^T = B^T A^T \tag{4}$$

This is easy to prove.

$$((AB)^T)_{ij} = (AB)_{ji} = \sum_k A_{jk} B_{ki} = \sum_k (B^T)_{ik} (A^T)_{kj} = (B^T A^T)_{ij} \tag{5}$$

A symmetric matrix is a matrix for which the transpose is itself. In other words,

$$M^T = M \tag{6}$$

Here is an example of a symmetric matrix.

$$A = \begin{pmatrix} 1 & -5 & 4 \\ -5 & 3 & 2 \\ 4 & 2 & 0 \end{pmatrix} \tag{7}$$

Notice that a symmetric matrix necessarily has the same number of rows and columns. In the above example, this number is 3.

In quantum mechanics, it is more useful to use the Hermitian conjugate (Or Conjugate Transpose) than the transpose. The Hermitian conjugate of a matrix A is defined as follows:

$$(A^\dagger)_{ij} = A^*_{ji} \tag{8}$$

Here, * denotes complex conjugate, and † is pronounced "dagger." Similarly as in the case of transpose, one can easily see the following relation:

$$(AB)^\dagger = B^\dagger A^\dagger \tag{9}$$

If the Hermitian conjugate of a matrix is itself, we call the matrix a Hermitian matrix. Here is an example of a Hermitian matrix:

$$A = \begin{pmatrix} 1 & 1+i & 2-3i \\ 1-i & 3 & 4 \\ 2+3i & 4 & 2 \end{pmatrix} \tag{10}$$

Notice that the diagonal part of a Hermitian matrix is necessarily real. (1, 3, and 2 are the diagonal parts. i.e. the top left to bottom right diagonal parts) This is expected, because of the following condition:

$$A^*_{kk} = A_{kk} \tag{11}$$

(By definition, a Hermitian matrix satisfies $A_{ij} = A^*_{ji}$. Set both i and j equal to k.)

Problem 1. Prove the followings:

$$(A + B)^\dagger = A^\dagger + B^\dagger \tag{12}$$

$$(A^\dagger)^\dagger = A \tag{13}$$

$$(ABC)^\dagger = C^\dagger B^\dagger A^\dagger \tag{14}$$

$$(cA)^\dagger = c^* A^\dagger \tag{15}$$

where c is a number.

Problem 2. Prove that $D + D^\dagger$ is Hermitian for any arbitrary square matrix D. (Hint[1])

Problem 3. A matrix E is called "anti-Hermitian" if it satisfies $E = -E^\dagger$. Prove that $F - F^\dagger$ is anti-Hermitian for any arbitrary square matrix F.

Problem 4. Prove that iG is anti-Hermitian if G is Hermitian. (Hint[2])

[1] Use (12) and (13)

[2] Use (15).

Problem 5. Prove that any arbitrary square matrix H can be expressed as a sum of Hermitian matrix and anti-Hermitian matrix.

Problem 6. Prove that $JJ^\dagger$ is Hermitian for any arbitrary square matrix J. (Hint[3])

Problem 7. Prove that $AB - BA$ is anti-Hermitian if A and B are Hermitian.

Problem 8. A matrix U is called "unitary" if it satisfies $UU^\dagger = I$. Prove that e^{iH} is unitary for any arbitrary Hermitian matrix H. (It may seem odd that an exponent can be a matrix. However, we can understand the expression in terms of Taylor series as follows.) (Hint[4])

$$e^{iH} = 1 + (iH) + \frac{(iH)^2}{2!} + \frac{(iH)^3}{3!} + \frac{(iH)^4}{4!} + \cdots \tag{17}$$

Summary

- The dagger $\dagger$ is defined by $(A^\dagger)_{ij} = A^*_{ji}$.
- A Hermitian matrix A satisfies $A^\dagger = A$.
- The diagonal part of a Hermitian matrix is necessarily real.
- $(A+B)^\dagger = A^\dagger + B^\dagger$
- $(AB)^\dagger = B^\dagger A^\dagger$
- $(cA)^\dagger = c^* A^\dagger$.

[3] Use (9) and (13).

[4] Show $(e^{iH})^\dagger = 1 + (-iH) + \frac{(-iH)^2}{2!} + \cdots$ and use the following expansion that is satisfied by any number c.

$$e^c e^{-c} = \left(1 + c + \frac{c^2}{2!} + \frac{c^3}{3!} + \cdots\right)\left(1 + (-c) + \frac{(-c)^2}{2!} + \frac{(-c)^3}{3!} + \cdots\right) = 1 \tag{16}$$