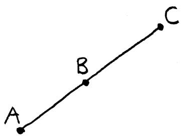
# The triangle inequality

The triangle inequality states that the length of any one sides of a triangle is always smaller than the sum of the lengths of the other two sides. I will assume that you are familiar with this inequality. If not, don't worry. I will prove it for you in this article.

Now, suppose you pick any three points $A$, $B$ and $C$ on a plane and connect these points to form a triangle. If a triangle is indeed formed, the triangle inequality will be satisfied (as we will show).

However, there are cases in which a triangle is not formed, even though you connect these points. This is when the three points lie on a line. See the figure.




If you count this object as a "triangle" even though you are not technically allowed to do so, you see that its longest "side" (i.e. $\overline{AC}$) is equal to the sum of the other two "sides" (i.e. $\overline{AB} + \overline{BC}$), and the triangle has 180°, 0° and 0° as its "angles." If you include this case, we can say, for any points $A$, $B$ and $C$, the distance between $A$ and $C$ is always less than or equal to the sum of the distance between $A$ and $B$ and the distance between $B$ and $C$. Notice that the phrase "equal to" is included in the above statement to account for the case depicted in the figure. In any case, if we denote the coordinate of $A$ as $(A_x, A_y)$ and similarly for $B$ and $C$, this statement can be translated into formulas as follows:

$$\sqrt{(C_x - A_x)^2 + (C_y - A_y)^2} \leq \sqrt{(B_x - A_x)^2 + (B_y - A_y)^2} + \sqrt{(C_x - B_x)^2 + (C_y - B_y)^2}$$

We can actually prove the above inequality by using the Cauchy-Schwarz inequality. First, let's set $B_x - A_x = X_1$, $B_y - A_y = Y_1$, $C_x - B_x = X_2$, and $C_y - B_y = Y_2$. Then, the above inequality can be re-written as:

$$\sqrt{(X_1 + X_2)^2 + (Y_1 + Y_2)^2} \leq \sqrt{X_1^2 + Y_1^2} + \sqrt{X_2^2 + Y_2^2}$$

Squaring both sides, and subtracting common terms we get: (**Problem 1.** Check this!)

$$X_1 X_2 + Y_1 Y_2 \leq \sqrt{X_1^2 + Y_1^2}\sqrt{X_2^2 + Y_2^2} \tag{1}$$

However, in our article on the Cauchy-Schwarz inequality, we proved the following:

$$(X_1X_2 + Y_1Y_2)^2 \leq (X_1^2 + Y_1^2)(X_2^2 + Y_2^2) \tag{2}$$

Taking the square root of both sides, we conclude (1) is correct. Thus, we have proved the triangle inequality.

**Problem 1.** We will consider now the three-dimensional version of our earlier problem in this article. Let's say that the coordinate of $A$ is $(A_x, A_y, A_z)$, the coordinate of $B$ is $(B_x, B_y, B_z)$ and the coordinate of $C$ is $(C_x, C_y, C_z)$. Using the following Cauchy-Schwarz inequality for three variables

$$(X_1X_2 + Y_1Y_2 + Z_1Z_2)^2 \leq (X_1^2 + Y_1^2 + Z_1^2)(X_2^2 + Y_2^2 + Z_2^2), \tag{3}$$

prove that the distance between $A$ and $C$ is less than equal to the sum of the distance between $A$ and $B$ and the distance between $B$ and $C$.

## Summary

- The triangle inequality can be proven by using the Cauchy-Schwarz inequality.