What is a vector?

Suppose you move 5 km North and 4 km East from your starting point. Then you further move 3 km South and 3 km East. See Fig. 1. What will be your final location? You will be at a position $2(= 5 - 3)$ km North and $7(= 4 + 3)$ km East from the starting point. We can express this relation more succinctly by using a concept called a "vector."

For example, if we express your first movement by $\vec{A}$ and your second movement by $\vec{B}$, then your total movement $\vec{C}$ can be expressed as follows:

$$\vec{C} = \vec{A} + \vec{B} \tag{1}$$

See Fig. 2. A notation like this is called "vector notation." $\vec{A}$, $\vec{B}$ and $\vec{C}$ are called vectors. You read $\vec{A}$ as "vector A." Some books use bold script to denote the vectors. For example, the above formula is often written as $\mathbf{C} = \mathbf{A} + \mathbf{B}$. Notice also that $\vec{B} + \vec{A} = \vec{A} + \vec{B}$, since you will be at the same position, if you perform the second movement $\vec{B}$ first and then the first movement $\vec{A}$. See Fig. 3

From now on, we will call the East direction the positive x-direction, the West direction the negative x-direction, the North direction the positive y-direction, and the South direction the negative y-direction. Now we can express vectors $\vec{A}$ and $\vec{B}$ as follows:

$$\vec{A} = 4\hat{x} + 5\hat{y}, \qquad \vec{B} = 3\hat{x} - 3\hat{y} \tag{2}$$

Here, $\hat{x}, \hat{y}$ are called unit vectors.
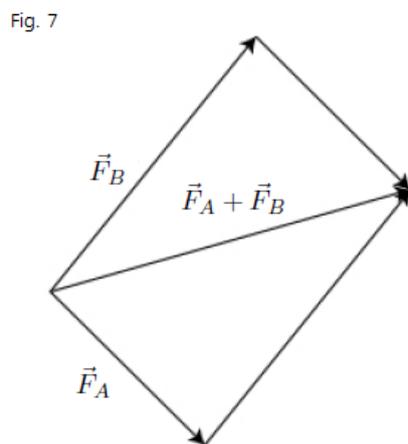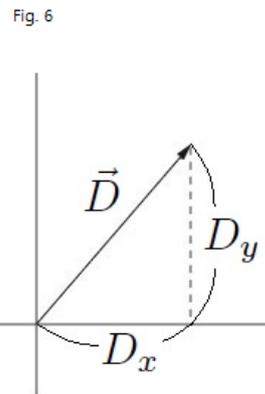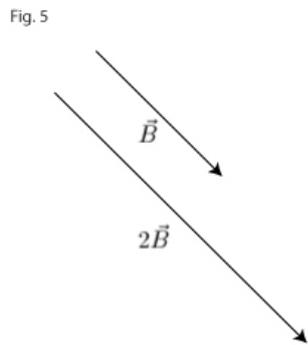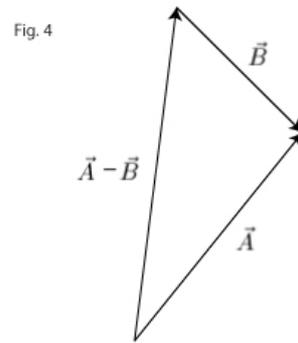
Then, it is easy to calculate their sum as follows:

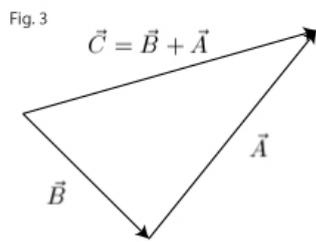$$\vec{C} = \vec{A} + \vec{B} = 4\hat{x} + 5\hat{y} + 3\hat{x} - 3\hat{y} = (4+3)\hat{x} + (5-3)\hat{y} = 7\hat{x} + 2\hat{y} \tag{3}$$

Expressed this way, it is easy to see again that $\vec{B} + \vec{A} = \vec{A} + \vec{B}$ should be satisfied. This is an example of vector addition. We can similarly perform vector subtraction,

$$\vec{A} - \vec{B} = 4\hat{x} + 5\hat{y} - (3\hat{x} - 3\hat{y}) = (4-3)\hat{x} + (5--3)\hat{y} = \hat{x} + 8\hat{y} \tag{4}$$

See Fig. 4. From the figure, we easily see the following:

$$(\vec{A} - \vec{B}) + \vec{B} = \vec{A} \tag{5}$$

Fig. 1

4km   3km

5km   3km

Fig. 2

$\vec{A}$
$\vec{B}$
$\vec{C} = \vec{A} + \vec{B}$

Fig. 3

$\vec{C} = \vec{B} + \vec{A}$
$\vec{A}$
$\vec{B}$

Fig. 4

$\vec{B}$
$\vec{A} - \vec{B}$
$\vec{A}$

Fig. 5

$\vec{B}$
$2\vec{B}$

Fig. 6

$\vec{D}$
D_y
D_x

Fig. 7

$\vec{F}_B$
$\vec{F}_A + \vec{F}_B$
$\vec{F}_A$
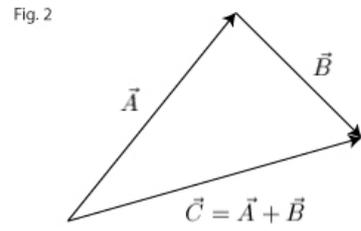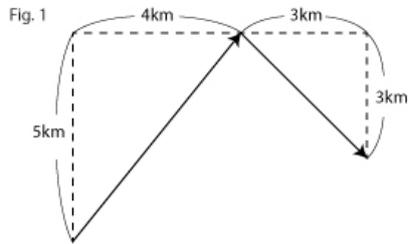
We can also multiply a vector by a scalar (i.e an ordinary number as opposed to a vector) as in the following examples:

$$2\vec{B} = 2(3\hat{x} - 3\hat{y}) = 6\hat{x} - 6\hat{y} \tag{6}$$

See Fig. 5.

We also want to note that a vector has a magnitude. See Fig. 6. By the Pythagorean theorem, the length of $\vec{D} = D_x\hat{x} + D_y\hat{y}$ is equal to $\sqrt{D_x^2 + D_y^2}$. The magnitude of a vector is defined by the length of vector. We can express this as:

$$|\vec{D}| = |D_x\hat{x} + D_y\hat{y}| = \sqrt{D_x^2 + D_y^2} \tag{7}$$

where $|\vec{D}|$ denotes the magnitude of $\vec{D}$. If you also look at Fig. 5, you will notice that the length of $2\vec{B}$ is the double of the length of $\vec{B}$. In other words, $|2\vec{B}| = 2|\vec{B}|$. You will be asked to prove a more general statement in Problem 5.

From our discussion so far it is also easy to see that a vector has a direction besides a magnitude. Remember that an ordinary number (a "scalar") doesn't have any direction. An example of a vector is velocity, while an example of a scalar is speed. 100km/h to the north direction is a velocity while 100km/h is just speed.

Notice also from our example of Fig. 5. that the direction of a vector doesn't change when you multiply it by a positive scalar; $2\vec{B}$ has the same direction as $\vec{B}$ because 2 is a positive scalar. However, when you multiply a vector by a negative scalar, the direction of the vector becomes the direct opposite. In either case, a vector multiplied by a non-zero scalar remains parallel to the original vector.

Another good example of a vector is force. For example, if two forces, $\vec{F}_A$ and $\vec{F}_B$ act on an object simultaneously, the total force exerted on the object is given by $\vec{F}_A + \vec{F}_B = \vec{F}_B + \vec{F}_A$. See Fig. 7. You see that the figure is slightly different from Fig. 2 or Fig. 3, since it is not the case that force $\vec{F}_A$ is exerted before force $\vec{F}_B$ is exerted. Nor vice versa, as they act simultaneously. Nevertheless, the additive result is the same. If $\vec{F}_A = 4\hat{x} + 5\hat{y}$, and $\vec{F}_B = 3\hat{x} - 3\hat{y}$, then $\vec{F}_A + \vec{F}_B = 7\hat{x} + 2\hat{y}$, just like $\vec{A} + \vec{B}$ was $7\hat{x} + 2\hat{y}$, when $\vec{A} = 4\hat{x} + 5\hat{y}$ and $\vec{B} = 3\hat{x} - 3\hat{y}$

All we said here can be generalized to higher-dimensions. For example, in 3-d, a vector can be expressed as follows:

$$\vec{V} = v_x\hat{x} + v_y\hat{y} + v_z\hat{z} \tag{8}$$

We would also like to remark that some textbooks use the following notations for the unit vectors;

$$\hat{i} = \hat{x}, \quad \hat{j} = \hat{y}, \quad \hat{k} = \hat{z} \tag{9}$$

Finally, let us conclude this article by introducing the concept of "relative velocity." Suppose an object moves in positive x-direction with speed 100 km/h, and you move in positive x-direction with speed 80 km/h. Then, at what velocity will you measure the object? Certainly, it will be 20 km/h in positive x-direction. You have obtained this velocity (denoted "$\vec{v}'$") by subtracting your velocity (denoted "$\vec{v}_0$") from the velocity of the object ("denoted $\vec{v}$") as follows:

$$\vec{v}' = \vec{v} - \vec{v}_0 \tag{10}$$

This is the concept of relative velocity. Note that the above equation is written in vector form. This suggests that this equation is not only valid when the object and you are moving in the same direction. For example, if another object moves northwards with 80 km/h and you move eastwards 80 km/h, you will see that the object moves north-westwards with speed $\sqrt{80^2 + 80^2}$ km/h by using Pythagorean theorem.

Problem 1. Can the velocity of a car change, while its speed remains the same? Explain! Can the speed of a car change, while its velocity remains the same? Explain!

Problem 2. Let $\vec{A}$ and $\vec{B}$ given by (2). Evaluate the following. (Hint[1])

$$|-\hat{x} + \hat{y}| =?, \qquad |4\hat{x} - 3\hat{y}| =?, \qquad |-10\hat{y}| =?$$

$$3\vec{A} - 4\vec{B} =?, \qquad |3\vec{A} - 4\vec{B}| =?$$

$$\vec{A} - 2\vec{B} =?, \qquad |\vec{A} - 2\vec{B}| =?$$

Problem 3. Let $\vec{D} = D_x\hat{x} + D_y\hat{y} + D_z\hat{z}$. What is the magnitude of this vector? Show in particular that the magnitude of this vector is non-negative, and is zero only when $\vec{D} = 0$ (i.e. when $\vec{D}$ is a zero vector). (Hint[2])

Problem 4. What is the magnitude of the vector $\vec{V} = 2\hat{x} - 2\hat{y} + \hat{z}$? (Hint: Use the result of Problem 3.)

Problem 5. Let $\vec{D} = D_x\hat{x} + D_y\hat{y} + D_z\hat{z}$. Prove $|c\vec{D}| = |c||\vec{D}|$ where c is a scalar. (Hint[3])

Problem 6. Suppose you move in the positive x-direction with a speed of 80km/h, and a car is moving in the negative y-direction with a speed of 60km/h. What is the speed of the car relative to you?

Problem 7. Using the triangle inequality, show that the following always holds for two arbitrary vectors $\vec{E}$, and $\vec{F}$. (Hint[4])

$$|\vec{E} + \vec{F}| \leq |\vec{E}| + |\vec{F}| \tag{11}$$

When does the equality hold?

---

[1]Use (7).

[2]You would need to use the 3-dimensional analog of the Pythagorean theorem. See our earlier article "Distances between points and equations for circles."

[3]Use $\sqrt{AB} = \sqrt{A}\sqrt{B}$ and $|c| = \sqrt{c^2}$.

[4]Draw $\vec{E}$, $\vec{F}$, and $\vec{E} + \vec{F}$. It will look like a triangle unless $\vec{E}$ and $\vec{F}$ are parallel.

Three comments. First comment. In this article, we have seen that a vector can be multiplied by a scalar. In later articles, we will see that a vector can be multiplied by another vector.

Second comment. In later articles, we will see that the pivotal objects in quantum mechanics are infinite-dimensional vector. Before I learned quantum mechanics, this sounded to me somewhat unnatural, and arbitrary. It was strange that they should be counted as vectors, if they are infinte-dimensional. Nevertheless, after learning quantum mechanics, they all made sense to me.

Third comment. We have seen that the magnitude of a vector is zero only when the vector is a zero vector. However, when we learn the theory of relativity by Einstein, we will see that a non-zero vector can have magnitude zero. Such a vector is called "light-like" vector. This concept was introduced not by Einstein, but by Einstein's former professor Minkowski.

Summary

- A vector is an object that has a direction besides magnitude.

- A vector can be added or subtracted by another vector.

- A vector can be multiplied by a scalar.

- The magnitude of a vector is defined by its length.

- The magnitude of a vector can be obtained by the Pythagorean theorem.