# What is a group? What is representation?

The concept of group is very important in physics and mathematics. In this article, we introduce the definition of a group and some examples by closely following wikipedia and *Lie Algebras in Particle Physics* by Howard Georgi. A group $G$ is a set with an operation $\bullet$ that combines any two elements $f$ and $g$ to form another element, denoted $f \bullet g$ or $fg$. To qualify as a group, the set and operation, $(G, \bullet)$, must satisfy following four requirements known as the group axiom.

**(Closure)** If $f$ and $g$ are in $G$ then $h = f \bullet g$ is always in $G$.
**(Associativity)** If $f$, $g$ and $h$ are in $G$ then $f \bullet (g \bullet h) = (f \bullet g) \bullet h$ is always satisfied.
**(Identity element)** There exists an element $e$ in $G$ such that for every element $f$ in $G$, $e \bullet f = f \bullet e = f$ is satisfied.
**(Inverse element)** For every element $f$ in $G$, there exists an inverse $f^{-1}$ such that $f \bullet f^{-1} = f^{-1} \bullet f = e$

A good example of a group is integer with the operation being addition. It satisfies the group axiom as follows.

**(Closure)** If $a$ and $b$ are integers then $a + b$ is always integer.
**(Associativity)** If $a$, $b$ and $c$ are integers, then $a + (b + c) = (a + b) + c$ is always satisfied.
**(Identity element)** For any integer $a$, $0 + a = a + 0 = a$ is satisfied. Therefore 0 is the identity element.
**(Inverse element)** For any integer $a$, there exists an inverse $-a$ such that $a + (-a) = (-a) + a = 0$

Now we want to represent a group using matrices. A representation of $G$ is a mapping, $D$ of the elements of $G$ onto a set of matrices with the following properties.

1. $D(e) = I$ where $I$ is the identity matrix.
2. $D(g_1)D(g_2) = D(g_1 \bullet g_2)$

As an aside, we want to remark that there is a field in mathematics called "representation theory." Apparently, mathematicians study how we can represent groups.

**Problem 1.** Show that the set of real number with the operation multiplication is not a group. Which axiom is violated? (Hint: the third axiom is not violated if you set the identity element to be 1.) Show that the set of real number without 0 and with multiplication as the group operation is a group.

**Problem 2.** Show that the group we considered as an example (i.e. integers with addition) can be represented as follows:

$$D(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \tag{1}$$

**Problem 3.** Show that $D(g) = I$ for any $g$ in group $G$ is a representation. Of course, it goes without saying that such a representation is useless. It is called "trivial representation," and exists for any group.

**Problem 4.** Show that the set of invertible $n \times n$ matrices forms a group if the group action $\bullet$ is defined to be ordinary matrix multiplication. Such a group is called "the general linear group of $n$" and denoted by $GL(n, \mathbf{R})$ if the entries in the matrix are real numbers and $GL(n, \mathbf{C})$ if the entries in the matrix are complex numbers.

**Problem 5.** Show that the set of $n \times n$ matrices with determinant 1 forms a group if the group action $\bullet$ is defined to be ordinary matrix multiplication. Such a group is called "the special linear group of $n$" and denoted by $SL(n, \mathbf{R})$ if the entries in the matrix are real numbers and $SL(n, \mathbf{C})$ if the entries in the matrix are complex numbers. As an aside $SL(2, \mathbf{C})$ describes what is called "Möbius transformation" (Yes, the same Möbius as in Möbius strip) and plays a very important role in string theory.

**Problem 6.** Show that $SU(N)$, which is the $N \times N$ unitary matrix with determinant 1, $SO(N)$,which is the $N \times N$ orthogonal matrix with determinant 1, and $SO(1, 3)$ form groups respectively. The latter is known as the Lorentz group for an obvious reason. We will talk more about it in our later article "The Lorentz group and its representations."

## Summary

- A group $G$ is a set with an operation $\bullet$ that combines any two elements $f$ and $g$ to form another element, denoted $f \bullet g$. A group needs to satisfy the group axiom.
- Closure, associativity, the existence of the identity element, the existence of inverse element form the group axiom.
- Matrices can be used to represent a group.